

# TWISTED ENDOSCOPY AND THE GENERIC PACKET CONJECTURE

BY

TAKUYA KONNO\*

*Graduate School of Mathematics, Kyushu University*

*812-8581 Hakozaki, Higashi-ku, Fukuoka, Japan*

*e-mail: takuya@math.kyushu-u.ac.jp*

*URL: <http://knmac.math.kyushu-u.ac.jp/T.Konno.html>*

## ABSTRACT

We prove a twisted analogue of the result of Rodier and Mœglin-Waldspurger on the dimension of the space of degenerate Whittaker vectors. This allows us to prove that certain twisted endoscopy for  $GL(n)$  implies the (local) generic packet conjecture for many classical groups.

## 1. Introduction

The Whittaker model of an irreducible representation was first introduced by Jacquet-Langlands as a natural local counter part of the Fourier coefficients of automorphic forms on  $GL(2)$ . Its extension to  $GL(n)$  was used extensively by Bernstein-Zelevinsky to study the irreducible representations of  $GL(n)$  over a  $p$ -adic field [4], [20], which later played a fundamental role in the study of the Rankin-Selberg  $L$ -functions of  $GL(n)$  by Jacquet-Piatetskii-Shapiro-Shalika. Their results suggest certain relationships between the representation theory of reductive groups over a local field and certain automorphic  $L$  and  $\varepsilon$ -factors, as was conjectured by Langlands.

Let  $F$  be a local field and  $G$  a connected reductive quasisplit group over  $F$ . Take a Borel subgroup  $\mathbf{B}$  of  $G$  defined over  $F$  and write  $\mathbf{U}$  for its unipotent radical. We

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can speak of the Whittaker model associated to a character  $\psi_{\mathbf{U}}$  of  $\mathbf{U}(F)$ , which is non-degenerate in the sense that its stabilizer in  $\mathbf{B}(F)$  equals  $Z_G(F)\mathbf{U}(F)$ . If an irreducible smooth representation  $\pi$  of  $G(F)$  is  $\psi_{\mathbf{U}}$ -**generic**, i.e., if it admits a Whittaker model, then Shahidi defined a large family of its corresponding automorphic  $L$  and  $\varepsilon$ -factors, and obtained the desired relationships [16]. Thus we might expect that his results can be extended to the representations which are not  $\psi_{\mathbf{U}}$ -generic for any  $\psi_{\mathbf{U}}$ .

Conjecturally the set  $\Pi_{\text{temp}}(G(F))$  of isomorphism classes of irreducible tempered representations of  $G(F)$  should be partitioned into a disjoint union of finite subsets  $\Pi_{\varphi}$ , called  **$L$ -packets** parametrized by the so-called Langlands parameters  $\varphi$  for  $G$ . The elements of  $\Pi_{\varphi}$  should share the same  $L$  and  $\varepsilon$ -factors which can be directly defined from  $\varphi$ . Thus to extend Shahidi's definition of Euler factors, it suffices to find a generic element in each  $\Pi_{\varphi}$ . This is exactly the assertion of the **generic packet conjecture**. Besides the case of archimedean  $F$  which was settled by Vogan [19], little is known about this conjecture.

From now on, we assume that  $F$  is non-archimedean. Then before discussing the generic packet conjecture, we must assume that the tempered  $L$ -packets are defined and satisfy some reasonable properties. At present, these assumptions are verified only for  $GL(n)$ ,  $SL(n)$  and  $U(3)$ . The conjecture in the  $GL(n)$  and  $SL(n)$  cases are due to Bernstein [20]. On the other hand, Gelbart-Rogawski-Soudry obtained a beautiful description of the endoscopic  $L$ -packets of  $U(3)$  in terms of theta liftings, and deduced the conjecture for them [7]. More recently, Friedberg-Gelbart-Jacquet-Rogawski established the generic packet conjecture for the rest (stable) tempered  $L$ -packets of  $U(3)$  by comparisons of the relative trace formulae [6]. Their results also include the global counter part of the conjecture. These approaches are different from the method of Vogan which relates the genericity to certain growth property of representations. To treat the general case, it is desirable to have a method similar to his. In fact, a result of Rodier [14] relates the genericity of an irreducible representation  $\pi$  with the asymptotic behavior of the distribution character  $\text{tr } \pi$  around the identity. If a tempered  $L$ -packet  $\Pi$  of  $G$  is endoscopic, we have a tempered  $L$ -packet  $\Pi^H$  of an endoscopic group  $H$  which lifts to  $\Pi$ . In this situation, Shahidi uses Rodier's result to reduce the generic packet conjecture for  $\Pi$  to that for  $\Pi^H$  [16, § 9].

In this paper, we shall examine an extended version of Shahidi's approach. We shall establish a certain twisted version of [14]. Then we apply this to the (still conjectural at present) twisted endoscopic lifting for  $GL(n)$  with respect to an outer automorphism  $\theta$  [2, § 9], which gives liftings of the tempered  $L$ -packets of

many classical groups to irreducible tempered representations of  $GL(n)$ . Then we deduce the generic packet conjecture for these classical groups from the existence of such twisted endoscopic liftings. Since Arthur's program on the twisted endoscopy for  $GL(n)$  is considered as the only realistic way to determine the  $L$ -packets (and Arthur packets) for classical groups, our result can be taken as an assertion that the generic packet conjecture is not far beyond the determination of tempered  $L$ -packets. On the other hand, because of the use of passage to Lie algebras, the restriction on the residual characteristic (it must not divide the order of  $\theta$ ) is inevitable.

Now we shall explain the ingredient of the paper. We shall prove a twisted analogue of the result of Mœglin and Waldspurger [13] rather than that of [14]. This facilitates us to consider more interesting examples of twisted characters of non-generic representations. In §2, we review the definition of the space of degenerate Whittaker vectors and specify the action of an automorphism  $\theta$  of finite order on it. To formulate the twisted analogue of the result of [13], we recall the local expansion of twisted characters [5] in §3. The twisted version is stated in §4 (Th. 4.1). Since [13] establishes some parts of Kawanaka's conjecture [10, Conj. (2.5.3)] on the relationships between degenerate Whittaker models and the wave front set of irreducible representations, we hope our result will shed some light on the conjecture also. This section also contains a few examples. The key ingredient of the proof is the explicit descent to  $G^\theta(F)$  of the test function  $\varphi_n$  of [13]. This is done in §6. Combining this with the infinitesimal construction of [13], which we review in §5, the proof of Th. 4.1 is completed in §7. In the final section §8, we combine our result with the conjectural character identity in the twisted endoscopy for  $GL(n)$  [2], and deduce the generic packet conjecture. The argument is quite similar to [16, § 9]. In the appendix, a classification of the elliptic twisted endoscopic data for  $GL(n)$  is given.

Although our method works only in the case  $G = U(3)$  at the moment, where the result is already known by [6], it should be applied to more wider class of groups once the twisted endoscopic lift is established. Degenerate Whittaker models were used by Mœglin to define the wave front sets for representations of  $p$ -adic groups. She also obtained many interesting results about this wave front set and its relationships with the theta correspondence. It should be interesting to consider the twisted analogues of these results and the behavior of the wave front sets under twisted endoscopic liftings. The examples contained in §4 already suggest some basic principle in this direction.

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## 2. Automorphisms and degenerate Whittaker models

Let  $G$  be a connected reductive group over a  $p$ -adic field  $F$  of odd residual characteristic. We fix a non-trivial additive character  $\psi$  of  $F$ , and a non-degenerate  $G(F)$ -invariant bilinear form  $B(\cdot, \cdot)$  on the Lie algebra  $\mathfrak{g}(F)$  of  $G(F)$ .

**2.1. DEGENERATE WHITTAKER MODELS.** Recall that a degenerate Whittaker model (or a generalized Gelfand-Graev model) is associated to a pair  $(N, \phi)$  [13, I.7] (cf. [10, 2.2]). Here  $N \in \mathfrak{g}(F)$  is a nilpotent element and  $\phi: \mathbb{G}_m \rightarrow G$  is an  $F$ -rational homomorphism such that

$$\mathrm{Ad}(\phi(t))N = t^{-2}N, \quad \forall t \in \mathbb{G}_m.$$

Let us recall the construction.  $\phi$  gives a gradation  $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ , where

$$\mathfrak{g}_i := \{X \in \mathfrak{g} \mid \mathrm{Ad}(\phi(t))X = t^i X, t \in \mathbb{G}_m\}.$$

If we write  $\mathfrak{g}^N$  for the centralizer of  $N$  in  $\mathfrak{g}$ , then we can take its  $\mathrm{Ad}(\phi(\mathbb{G}_m))$ -stable complement  $\mathfrak{m}$  in  $\mathfrak{g}$ :  $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}^N$ . The above gradation restricts to gradations  $\mathfrak{m} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{m}_i$  and  $\mathfrak{g}^N = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i^N$ . We introduce two unipotent subgroups

$$U := \exp\left(\sum_{i \geq 1} \mathfrak{g}_i\right), \quad V := \exp\left(\mathfrak{g}_1^N + \sum_{i \geq 2} \mathfrak{g}_i\right).$$

The character

$$\chi: V(F) \ni \exp X \mapsto \psi(B(N, X)) \in \mathbb{C}^1$$

is well-defined and stable under  $\mathrm{Ad}(U(F))$  [13, I.7 (2)].

$$B_N(X, Y) := B(N, [X, Y]) \quad (X, Y \in \mathfrak{g})$$

restricts to a non-degenerate alternating form on  $\mathfrak{m}$ . In fact it is a duality between  $\mathfrak{m}_i$  and  $\mathfrak{m}_{2-i}$  for  $i \neq 1$  and an alternating form on  $\mathfrak{m}_1$ . Define  $\mathcal{H}_N = \mathcal{H}_{N, \phi}$  to be the Heisenberg group associated to  $(\mathfrak{m}_1(F), \psi \circ B_N)$  if  $\mathfrak{m}_1$  is not trivial, and  $\mathbb{C}^1 := \{z \in \mathbb{C} \mid z\bar{z} = 1\}$  otherwise.  $(\rho_N, \mathcal{S}_N)$  denotes the unique irreducible smooth representation of  $\mathcal{H}_N$  on which the center  $\mathbb{C}^1$  acts by the identity representation (multiplication). We have a homomorphism

$$p_N: U(F) \ni \exp X \mapsto (X_1^{\mathfrak{m}}; \psi(B(N, X))) \in \mathcal{H}_N,$$

where  $X_1^{\mathfrak{m}}$  is the  $\mathfrak{m}_1(F)$ -component of  $X$  under the decomposition  $\mathrm{Lie} U = \mathfrak{m}_1 \oplus \mathfrak{g}_1^N \oplus \sum_{i \geq 2} \mathfrak{g}_i$ . Write  $\bar{\rho}_N := \rho_N \circ p_N$ .

Now let  $(\pi, E)$  be an admissible representation of  $G(F)$ . We have the twisted coinvariant space [3]

$$E_{V,\chi} := E/E(V, \chi),$$

$$E(V, \chi) := \mathrm{Span}\{\pi(v)\xi - \chi(v)\xi \mid v \in V(F), \xi \in E\}.$$

Clearly,  $U(F)$  acts by some copy of  $\bar{\rho}_N$  on  $E_{V,\chi}$ . Define the space of **degenerate Whittaker vectors** with respect to  $(N, \phi)$  by

$$\mathcal{W}_{N,\phi}(\pi) := \mathrm{Hom}_{U(F)}(\bar{\rho}_N, E_{V,\chi}).$$

**2.2. ACTION OF AUTOMORPHISMS.** Let  $\theta$  be an  $F$ -automorphism of finite order  $\ell$  of  $G$ . Then  $\theta$  is automatically quasi-semisimple and a theorem of Steinberg shows that  $G^\theta$  is reductive. Write  $G^+$  for the (non-connected) reductive group  $G \rtimes \langle \theta \rangle$ . We impose, from now on, that  $B(\cdot, \cdot)$  is  $\theta$ -invariant,  $N \in \mathfrak{g}^\theta(F)$  and  $\phi$  is  $G^\theta$ -valued. The complement  $\mathfrak{m}$  of  $\mathfrak{g}^N$  can be chosen (and we do choose) to be  $\theta$ -stable. Following [1], we write  $\Pi(G(F)\theta)$  for the set of isomorphism classes of irreducible admissible representations of  $G^+(F)$  whose restrictions to  $G(F)$  are still irreducible. Also write  $\Xi$  for the group of characters of  $\pi_0(G^+) = \langle \theta \rangle$ .

Since  $\mathcal{H}_N$  is  $\theta$ -stable (we let  $\theta$  act trivially on the center  $\mathbb{C}^1$ ), we may form  $\mathcal{H}_N^+ := \mathcal{H}_N \rtimes \langle \theta \rangle$ .  $\rho_N$  is also  $\theta$ -stable and we fix an isomorphism  $\rho_{N,+}(\theta) : \theta(\rho_N) \xrightarrow{\sim} \rho_N$  satisfying  $\rho_{N,+}(\theta)^\ell = \mathrm{id}$ . This extends  $\rho_N$  to an irreducible representation  $\rho_{N,+}$  of  $\mathcal{H}_N^+$ .

Take  $(\pi, E) \in \Pi(G(F)\theta)$ . On the space  $\mathcal{W}_{N,\phi}(\pi)$  of degenerate Whittaker vectors for  $\pi|_{G(F)}$ , we define the action of  $\theta$  by

$$\theta(\Phi) : \mathcal{S}_N^{\rho_{N,+}(\theta)^{-1}} \xrightarrow{\Phi} \mathcal{S}_N \xrightarrow{\Phi} E_{V,\chi} \xrightarrow{\pi(\theta)} E_{V,\chi}.$$

Set  $U^+ := U \rtimes \langle \theta \rangle$  and extend  $p_N$  to  $U^+(F) \rightarrow \mathcal{H}_N^+$  by  $p_N(\theta) = \theta$ . If we write  $\rho_N^+ := \mathrm{Ind}_{\mathcal{H}_N}^{\mathcal{H}_N^+} \rho_N$  and  $\bar{\rho}_N^+ := \rho_N^+ \circ p_N$ , then the Mackey theory  $\rho_N^+ \simeq \bigoplus_{\zeta \in \Xi} \zeta \rho_{N,+}$  gives

$$\begin{aligned} \mathcal{W}_{N,\phi}(\pi) &\simeq \mathrm{Hom}_{U(F)}(E_{V,\chi}^*, \bar{\rho}_N^\vee) \simeq \mathrm{Hom}_{U^+(F)}(E_{V,\chi}^*, \mathrm{Ind}_{U(F)}^{U^+(F)} \bar{\rho}_N^\vee) \\ &\simeq \mathrm{Hom}_{U^+(F)}(\bar{\rho}_N^+, E_{V,\chi}) \simeq \bigoplus_{\zeta \in \Xi} \mathrm{Hom}_{U^+(F)}(\zeta \bar{\rho}_{N,+}, E_{V,\chi}). \end{aligned}$$

Here  $\pi^*$  denotes the dual of  $\pi$  while  $\pi^\vee$  is its contragredient. Putting  $\mathcal{W}_{N,\phi}(\pi)_\zeta := \mathrm{Hom}_{U^+(F)}(\zeta \bar{\rho}_{N,+}, E_{V,\chi})$ , we have  $\theta(\Phi) = \zeta(\theta)\Phi$  for  $\Phi \in \mathcal{W}_{N,\phi}(\pi)_\zeta$  and hence

$$(2.1) \quad \mathrm{tr}(\theta|\mathcal{W}_{N,\phi}(\pi)) = \sum_{\zeta \in \Xi} \zeta(\theta) \dim \mathcal{W}_{N,\phi}(\pi)_\zeta.$$

### 3. Local expansion of twisted characters

Let us recall Clozel's result on the singular behavior of twisted characters [5]. To state the result, we need Harish-Chandra's descent in the twisted case [*loc. cit.* 3.4].

3.1. DESCENT IN THE TWISTED CASE. We shall be concerned with the distributions invariant under the  $\theta$ -conjugation:

$$\mathrm{Ad}_\theta(g)x := gx\theta(g)^{-1}, \quad g, x \in G(F).$$

Writing  $\mathfrak{g}(\theta) := (1 - \theta)\mathfrak{g} \subset \mathfrak{g}$ , set

$$\begin{aligned} G^{\theta'}(F) &:= \{g \in G^\theta(F) \mid \det(\mathrm{Ad}(g) \circ \theta - 1|_{\mathfrak{g}(\theta, F)}) \neq 0\}, \\ \Omega_\theta &:= \mathrm{Ad}_\theta(G(F))G^{\theta'}(F). \end{aligned}$$

$\Omega_\theta$  is a dense,  $\mathrm{Ad}_\theta(G(F))$ -invariant subset of  $G(F)$ . Moreover, the map

$$G(F) \times G^{\theta'}(F) \ni (g, m) \longmapsto \mathrm{Ad}_\theta(g)m \in G(F)$$

is submersive [15, Prop. 1]. Then [8, Th. 11] asserts the existence of a surjective linear map

$$C_c^\infty(G(F) \times G^{\theta'}(F)) \ni \alpha(g, m) \longmapsto \varphi(x) \in C_c^\infty(\Omega_\theta)$$

such that

$$\int_{G(F)} \int_{G^\theta(F)} \alpha(g, m) \Phi(gm\theta(g)^{-1}) dm dg = \int_{G(F)} \varphi(x) \Phi(x) dx$$

for any  $\Phi \in C_c^\infty(G(F))$  and fixed invariant measures on  $G(F)$  and  $G^\theta(F)$ . We write  $\mathcal{D}(X)$  for the space of distributions on a  $\ell$ -space  $X$  in the sense of [3]. Dual to the above map is

$$\mathcal{D}(\Omega_\theta) \ni T \longmapsto \tau \in \mathcal{D}(G(F) \times G^{\theta'}(F))$$

given by  $\langle \tau, \alpha \rangle := \langle T, \varphi \rangle$ . Of course the map  $\alpha \mapsto \varphi$  is not injective, but

$$\phi(m) := \int_{G(F)} \alpha(g, m) dg \in C_c^\infty(G^{\theta'}(F))$$

is uniquely determined by  $\varphi$ .

LEMMA 3.1: For any  $\mathrm{Ad}_\theta(G(F))$ -invariant  $T \in \mathcal{D}(\Omega_\theta)$ , there exists an  $\mathrm{Ad}(G^\theta(F))$ -invariant distribution  $\sigma_T \in \mathcal{D}(G^{\theta'}(F))$  such that

$$\langle T, \varphi \rangle = \langle \sigma_T, \phi \rangle, \quad \forall \varphi \in C_c^\infty(\Omega_\theta).$$

In the ordinary case, this is [8, Lem. 21]. The word-to-word translation to the present case is easy.

3.2. LOCAL EXPANSION. Now let  $(\pi, E) \in \Pi(G(F)\theta)$ . The distribution

$$\Theta_{\pi, \theta}(f) := \mathrm{tr}(\pi(f) \circ \pi(\theta)), \quad f \in C_c^\infty(G(F))$$

is called the **twisted character** of  $\pi$ . This is clearly  $\mathrm{Ad}_\theta(G(F))$ -invariant and we can apply Lemma 3.1 to have an invariant distribution  $\vartheta_\pi$  on  $G^{\theta'}(F)$  such that

$$\Theta_{\pi, \theta}(\varphi) = \vartheta_\pi(\phi), \quad \forall \varphi \in C_c^\infty(\Omega_\theta).$$

Clozel showed that this  $\vartheta_\pi$  is “close to being admissible”. In particular, we have

THEOREM 3.2 ([5] Th. 3): Write  $\mathcal{N}(\mathfrak{g}^\theta(F))$  for the set of nilpotent  $\mathrm{Ad}(G^\theta(F))$ -orbits in  $\mathfrak{g}^\theta(F)$ . Then there exist a neighborhood  $U_{\pi, \theta}$  of 0 in  $\mathfrak{g}^\theta(F)$  and complex numbers  $c_{\mathfrak{o}, \theta}(\pi)$ ,  $\mathfrak{o} \in \mathcal{N}(\mathfrak{g}^\theta(F))$  such that

$$(3.1) \quad \Theta_{\pi, \theta}(\varphi) = \sum_{\mathfrak{o} \in \mathcal{N}(\mathfrak{g}^\theta(F))} c_{\mathfrak{o}, \theta}(\pi) \widehat{\mu}_\mathfrak{o}(\phi \circ \exp)$$

holds for any  $\varphi \in C_c^\infty(\Omega_\theta)$  with  $\mathrm{supp}(\phi \circ \exp) \subset U_{\pi, \theta}$ . Here we have fixed invariant measures  $dg$  and  $d\mu_\mathfrak{o}(X)$  on  $G(F)$  and  $\mathfrak{o} \in \mathcal{N}(\mathfrak{g}^\theta(F))$ , respectively.  $\widehat{\mu}_\mathfrak{o}$  denotes the Fourier transform of the orbital integral on  $\mathfrak{o}$ :

$$\widehat{\mu}_\mathfrak{o}(\phi \circ \exp) := \int_{\mathfrak{o}} (\widehat{\phi \circ \exp})(X) d\mu_\mathfrak{o}(X),$$

where

$$(\widehat{\phi \circ \exp})(X) := \int_{\mathfrak{g}^\theta(F)} \phi(\exp Y) \psi(B(X, Y)) dY.$$

Remark 3.3: The constants  $c_{\mathfrak{o}, \theta}(\pi)$  are unique up to factors determined by the choice of invariant measures. We follow the choice made in [13, I.8]. That is, we fix a self-dual invariant measures  $dX$  and  $dY$  on  $\mathfrak{g}(F)$  and  $\mathfrak{g}^\theta(F)$  with respect to  $\psi \circ B$ , respectively. By  $B_N$ , the tangent space  $T_N \mathfrak{o}$  of  $\mathfrak{o} \in \mathcal{N}(\mathfrak{g}^\theta(F))$  at  $N \in \mathfrak{o}$  is identified with  $\mathfrak{g}^\theta(F)/\mathfrak{g}^{N, \theta}(F) \simeq \mathfrak{m}^\theta(F)$ . Then we fix a self-dual invariant measure with respect to  $\psi \circ B_N$  on it. This determines an invariant measure  $\mu_\mathfrak{o}$  on  $\mathfrak{o}$ . Finally, we fix an invariant measure  $dg$  on  $G(F)$  so that the absolute value of the Jacobian of  $\exp$  relative to  $dX$  and  $dg$  at 0 is 1.

#### 4. A twisted analogue of the result of [13]

4.1. THE RESULT. Now we can state our main result. Let  $(\pi, E) \in \Pi(G(F)\theta)$ . Then we have the character expansion (3.1) and the ordinary character expansion at the identity [9, Th. 5]

$$(4.1) \quad \Theta_\pi(\varphi) = \sum_{\mathfrak{D} \in \mathcal{N}(\mathfrak{g}(F))} c_{\mathfrak{D}}(\pi) \widehat{\mu}_{\mathfrak{D}}(\varphi \circ \exp).$$

As in [13], we write  $\mathcal{N}_B(\pi)$  for the set of  $\mathfrak{D} \in \mathcal{N}(\mathfrak{g}(F))$  such that  $c_{\mathfrak{D}}(\pi) \neq 0$ . Similarly,  $\mathcal{N}_{B,\theta}(\pi)$  denotes the set of  $\mathfrak{o} \in \mathcal{N}(\mathfrak{g}^\theta(F))$  such that  $c_{\mathfrak{o},\theta}(\pi) \neq 0$ . We have a partial order  $\mathfrak{D} \geq \mathfrak{D}'$  on  $\mathcal{N}(\mathfrak{g}(F))$  defined by  $\overline{\mathfrak{D}} \supset \overline{\mathfrak{D}'}$ . Write  $\mathcal{N}_B(\pi)^{\max}$  for the set of maximal elements of  $\mathcal{N}_B(\pi)$  with respect to this order. We have a similar order on  $\mathcal{N}(\mathfrak{g}^\theta(F))$ , and the subset  $\mathcal{N}_{B,\theta}(\pi)^{\max}$  of maximal elements in  $\mathcal{N}_{B,\theta}(\pi)$ . But what we need below is the set  $\mathcal{N}_B^\theta(\pi)^{\max}$  of  $\mathfrak{o} \in \mathcal{N}_{B,\theta}(\pi)$  such that the  $\text{Ad}(G(F))$ -orbit  $\mathfrak{D} \in \mathcal{N}(\mathfrak{g}(F))$  containing it belongs to  $\mathcal{N}_B(\pi)^{\max}$ .

Again following [13], we write  $\mathcal{N}_{\text{Wh}}(\pi)$  for the set of  $\mathfrak{D} \in \mathcal{N}(\mathfrak{g}(F))$  such that  $\mathcal{W}_{N,\phi}(\pi) \neq 0$  for  $N \in \mathfrak{D}$  and a suitable choice of  $\phi$ . The subset of maximal elements  $\mathcal{N}_{\text{Wh}}(\pi)^{\max}$  is also defined. Let us write  $\mathcal{N}_{\text{Wh}}^\theta(\pi)^{\max}$  for the set of  $\mathfrak{o} \in \mathcal{N}(\mathfrak{g}^\theta(F))$  such that

(1) the element  $\mathfrak{D} \in \mathcal{N}(\mathfrak{g}(F))$  containing  $\mathfrak{o}$  belongs to  $\mathcal{N}_{\text{Wh}}(\pi)^{\max}$ ;

(2) for some  $N \in \mathfrak{o}$  and a suitable  $\phi$ ,  $\mathcal{W}_{N,\phi}$  is  $\theta$ -stable and  $\text{tr}(\theta|\mathcal{W}_{N,\phi}(\pi)) \neq 0$ .

By [13, I.16, 17] (1) assures that  $\dim \mathcal{W}_{N,\phi}(\pi) = c_{\mathfrak{D}}(\pi)$  is finite. Thus the condition (2) makes sense.

**THEOREM 4.1:** *Suppose  $\ell$  is prime to the residual characteristic of  $F$ , which we assume to be odd.*

(1)  $\mathcal{N}_{\text{Wh}}^\theta(\pi)^{\max} = \mathcal{N}_B^\theta(\pi)^{\max} \subset \mathcal{N}_{B,\theta}(\pi)^{\max}$ .

(2) Let  $\mathfrak{o} \in \mathcal{N}_B^\theta(\pi)^{\max}$ . Then for any choice of  $N \in \mathfrak{o}$ ,  $\phi$  and  $\mathfrak{m}$  as in §2.2, we have  $\text{tr}(\theta|\mathcal{W}_{N,\phi}(\pi)) = c_{\mathfrak{o},\theta}(\pi)$  modulo  $\ell$ -th roots of unity.

**Remark 4.2:** In (2) above, the ambiguity of  $\ell$ -th roots of unity occurs because our choice of the extension  $\rho_{N,+}$  of  $\rho_N$  is arbitrary. In the proof of this theorem, we shall adopt a particular choice of  $\rho_{N,+}$  (see §5.1 below) and prove the exact equality with that choice. In particular, if  $(N, \phi)$  is chosen so that  $\mathfrak{m}_1 = 0$ , we have the equality without any ambiguity.

4.2. EXAMPLES. Let us look at some very basic examples of the above theorem. First we consider the case of  $G_n := \text{Res}_{E/F} GL(n)$  for a finite cyclic extension  $E/F$ . Take the automorphism  $\theta$  to be a composite  $\theta_1 \circ \tilde{\sigma}$ , where  $\theta_1$



is any automorphism of  $GL(n)_E$  of finite order and  $\tilde{\sigma}$  is the  $F$ -automorphism of  $\text{Res}_{E/F} GL(n)$  associated to a generator  $\sigma$  of  $\text{Gal}(E/F)$ .

Recall the Zelevinsky classification of irreducible admissible representations of  $G_n(F)$ . A **segment**  $\Delta_{a,m}$  is the sequence  $[a+1, a+2, \dots, a+m]$  of integers.  $|\Delta_{a,m}| := m$  is the **length** of  $\Delta_{a,m}$ . For an irreducible supercuspidal representation  $\rho$  of  $G_d(F)$  and a segment  $\Delta_{a,m}$ , we write  $\langle \Delta_{a,m} \rangle_\rho$  for the unique irreducible subrepresentation of the parabolically induced representation

$$\text{ind}_{P_{(d^m)}(F)}^{G_{dm}(F)} [(\rho | \det_E^{a+1} \otimes \rho | \det_E^{a+2} \otimes \dots \otimes \rho | \det_E^{a+m}) \otimes \mathbf{1}_{U_{(d^m)}(F)}].$$

Here  $P_{(d^m)}$  denotes the standard parabolic subgroup of  $G_{dm}$  associated to the partition  $(d^m) = (d, d, \dots, d)$  ( $m$ -tuple) and  $U_{(d^m)}$  is its unipotent radical. A **multi-segment** is a finite sequence  $\Delta_{\mathbf{a}, \mathbf{m}} = [\Delta_{a^1, m^1}, \dots, \Delta_{a^r, m^r}]$  of segments ( $\mathbf{a} = \{a^i\}_{1 \leq i \leq r}$ ,  $\mathbf{m} = \{m^i\}_{1 \leq i \leq r}$ ) satisfying  $a_i \geq a_j$  for any  $1 \leq i < j \leq r$ . Then for an irreducible supercuspidal representation  $\rho$  of  $G_d(F)$ , the parabolically induced representation

$$\text{ind}_{P_{d\mathbf{m}}}^{G_{d|\mathbf{m}|}} [(\langle \Delta_{a^1, m^1} \rangle_\rho \otimes \dots \otimes \langle \Delta_{a^r, m^r} \rangle_\rho) \otimes \mathbf{1}_{U_{d\mathbf{m}}(F)}]$$

admits a unique irreducible subrepresentation  $\langle \Delta_{\mathbf{a}, \mathbf{m}} \rangle_\rho$ . Here  $|\mathbf{m}| = \sum_{i=1}^r m^i$  and  $d\mathbf{m}$  denotes its partition  $(dm^1, \dots, dm^r)$ . Now the Zelevinsky classification can be stated as follows.

**PROPOSITION 4.3** ([20] Th. 6.1, 7.1): (i) Take a finite family of pairs  $(\Delta_{\mathbf{a}_j, \mathbf{m}_j}, \rho_j)_{1 \leq j \leq s}$ , where  $\Delta_{\mathbf{a}_j, \mathbf{m}_j}$  is a multi-segment and  $\rho_j$  is an irreducible supercuspidal representation of  $G_{d_j}(F)$  ( $1 \leq j \leq s$ ) satisfying  $\rho_i \not\cong \rho_j$ ,  $i \neq j$ . Write  $\mathbf{n} = (n_1, \dots, n_s)$  with  $n_j := |\mathbf{m}_j|$  and  $n := |\mathbf{n}|$ . Then the parabolically induced module

$$\langle (\Delta_{\mathbf{a}_j, \mathbf{m}_j}, \rho_j)_j \rangle := \text{ind}_{P_{\mathbf{n}}}^{G_n(F)} [(\langle \Delta_{\mathbf{a}_1, \mathbf{m}_1} \rangle_{\rho_1} \otimes \dots \otimes \langle \Delta_{\mathbf{a}_s, \mathbf{m}_s} \rangle_{\rho_s}) \otimes \mathbf{1}_{U_{\mathbf{n}}(F)}]$$

is irreducible.

(ii) Each irreducible admissible representation  $\pi$  of  $G_n(F)$  is of the form  $\langle (\Delta_{\mathbf{a}_j, \mathbf{m}_j}, \rho_j)_j \rangle$ . The family  $(\Delta_{\mathbf{a}_j, \mathbf{m}_j}, \rho_j)_{1 \leq j \leq s}$  is uniquely determined by  $\pi$  up to permutations.

Let  $P$  be a parabolic subgroup of a connected reductive group  $G$  and  $\mathfrak{O}$  be a nilpotent  $M$ -orbit in the Lie algebra  $\mathfrak{m}$  of a Levi component  $M$  of  $P$ . Then the parabolically induced nilpotent  $G$ -orbit  $i_M^G(\mathfrak{O})$  in  $\mathfrak{g}$  was defined by Lusztig-Spaltenstein [18, II.3]. A nilpotent orbit parabolically induced from the zero orbit  $i_M^G(\{0\})$  is called the **Richardson orbit** from  $P$ . It is known that the

nilpotent orbits of  $\mathfrak{g}_n = \text{Lie } G_n$  are all Richardson orbits. In fact, if a nilpotent orbit  $\mathfrak{O}$  has the elementary divisors  $(T-1)^{n_1}, (T-1)^{n_2}, \dots, (T-1)^{n_s}$ , then it is the Richardson orbit  $i_{t_n}^{G_n}(\{0\})$  from  $P_{t_n}$ . Here  $t_n$  is the partition corresponding to the transpose of the Young diagram associated to  $n$  (the partition **dual** to  $n$ ). The following is a recapitulation of [13, Prop. II.2].

**PROPOSITION 4.4:** (i)  $\mathcal{N}_{\text{Wh}}(\langle(\Delta_{\mathbf{a}_j, \mathbf{m}_j}, \rho_j)_j\rangle)^{\max} = \mathcal{N}_B(\langle(\Delta_{\mathbf{a}_j, \mathbf{m}_j}, \rho_j)_j\rangle)^{\max}$  consists of the  $G_n(F)$ -orbit  $i_{(\mathbf{m}_1^{d_1}, \dots, \mathbf{m}_s^{d_s})}^{G_n}(\{0\})$ . Here  $\mathbf{m}^d$  denotes the partition  $(m_1^d, \dots, m_t^d)$  for  $\mathbf{m} = (m_1, \dots, m_t)$ .

(ii) For  $N \in i_{(\mathbf{m}_1^{d_1}, \dots, \mathbf{m}_s^{d_s})}^{G_n}(\{0\})$ , the space  $\mathcal{W}_{N, \phi}(\langle(\Delta_{\mathbf{a}_j, \mathbf{m}_j}, \rho_j)_j\rangle)$  is one dimensional.

Applying Th. 4.1 (2) to this, we have the following.

**COROLLARY 4.5:** Suppose an  $F$ -automorphism  $\theta$  of  $G_n$  has the finite order prime to the residual characteristic of  $F$ . Then for each  $\pi \in \Pi(G_n(F)\theta)$ , the twisted character  $\Theta_{\pi, \theta}$  is not zero.

Similarly we deduce the following from the uniqueness of (non-degenerate) Whittaker models [17].

**COROLLARY 4.6:** Suppose an  $F$ -automorphism  $\theta$  of a connected reductive quasisplit  $F$ -group  $G$  has the finite order prime to the residual characteristic of  $F$ . Let  $\chi$  be a character of the unipotent radical  $U(F)$  of an  $F$ -Borel subgroup  $B(F)$  such that  $\text{Stab}(\chi, B(F)) = Z_G(F)U(F)$ . Then the twisted character  $\Theta_{\pi, \theta}$  of an irreducible  $\chi$ -generic representation  $\pi$  is not trivial.

*Remark 4.7:* Both of these follow from the fact that the corresponding space  $\mathcal{W}_{N, \phi}(\pi)$  is one dimensional. Otherwise, even if  $\mathcal{W}_{N, \phi}(\pi) \neq 0$  the trace of  $\theta$  on it can be zero. An example of this phenomenon is constructed by Ju-Lee Kim and Piatetskii-Shapiro [11].

## 5. Infinitesimal construction of Mœglin and Waldspurger

Here we recall certain constructions from [13]. First we prepare some lattices in  $\mathfrak{g}(F)$ .

**5.1. LATTICES AND ESTIMATION FORMULAE.** We write  $\mathcal{O}$ ,  $\mathfrak{p}$ ,  $q_F$  and  $|\cdot|_F$  for the ring of integers of  $F$ , its unique maximal ideal, the cardinality of the residue field  $\mathcal{O}/\mathfrak{p}$  and the absolute value on  $F$  normalized as usual, respectively. We also fix a uniformizer  $\varpi$  of  $\mathcal{O}$ . We may assume that  $\psi$  fixed in §2 is of order zero. (This affects the statement only by multiplying some scalar to  $B(\cdot, \cdot)$ )

For  $G$  and  $\theta$ , we can take an  $F$ -group embedding  $\iota: G \hookrightarrow GL(n)_F$  and a semisimple element  $\tilde{\theta} \in GL(n, F)$  such that  $\iota \circ \theta = \text{Ad}(\tilde{\theta}) \circ \iota$ . Since  $\theta$  is of finite order, we may choose  $\tilde{\theta}$  in  $GL(n, \mathcal{O})$ . Write  $\Lambda^\iota$  for the lattice  $\iota^{-1}(\mathfrak{gl}(n, \mathcal{O}))$  in  $\mathfrak{g}(F)$ . Obviously  $\Lambda^\iota$  is  $\theta$ -stable and satisfies  $[\Lambda^\iota, \Lambda^\iota] \subset \Lambda^\iota$ . Taking  $A' \in \mathbb{N}$  sufficiently large, we may assume that  $\Lambda := \varpi^{A'} \Lambda^\iota$  satisfies  $B(X, Y) \in \mathcal{O}$ , for any  $X, Y \in \Lambda$ .

Let  $p$  be the residual characteristic of  $F$ . We write  $[F: \mathbb{Q}_p] = ef$  where  $e$  is the order of ramification and  $f$  is the modular degree of  $F$  over  $\mathbb{Q}_p$ , respectively. One sees immediately that

$$\text{ord}_F(a!) \leq \frac{ae}{p-1}.$$

Let  $A \in \mathbb{N}$  be such that  $\exp|_{\varpi^A \Lambda}$  is injective. For  $c \in \mathbb{N}$ , we replace  $A$  by  $A_1 := \sup(A, \frac{3e}{p-1} + c + 2)$  to have [13, I.1]

$$(5.1) \quad \forall X \in \varpi^n \Lambda, \forall Y \in \varpi^m \Lambda, \text{ with } n, m \geq A_1$$

$$\log(\exp X \exp Y) - (X + Y + \frac{1}{2}[X, Y]) \in \varpi^{n+m+c} \Lambda,$$

$$(5.2) \quad \forall X \in \varpi^n \Lambda \text{ with } n \geq A_1, \forall Y \in \varpi^m \Lambda$$

$$\text{Ad}(\exp X)Y - (Y + \text{ad}(X)Y) \in \varpi^{2n+m} \Lambda.$$

Recall that  $\psi \circ B_N$  gives a duality of  $\mathfrak{m}(F)$  with itself. As in [13, I.2], we introduce a lattice  $\Lambda' := \mathfrak{m}^{\Lambda'} \oplus \sum_i \Lambda \cap \mathfrak{g}_i^N(F)$ . Here  $\mathfrak{m}^{\Lambda'}$  is a certain lattice in  $\mathfrak{m}(F)$  which is self dual with respect to  $\psi \circ B_N$  and compatible with the gradation  $\mathfrak{m} = \bigoplus_i \mathfrak{m}_i$ . Fix  $d \in \mathbb{N}$  such that  $\varpi^d \Lambda \subset \Lambda' \subset \varpi^{-d} \Lambda$ . We can deduce [*loc. cit.* I.3] the following from (5.1) and (5.2). Fix  $C \geq 2d$ . For  $D \geq \sup(A_1 + d, C + 3d)$  we have:

$$(5.3) \quad \forall X \in \varpi^n \Lambda', \forall Y \in \varpi^m \Lambda', \text{ with } n, m \geq D$$

$$\log(\exp X \exp Y) - (X + Y + \frac{1}{2}[X, Y]) \in \varpi^{n+m+C} \Lambda,$$

$$(5.4) \quad \forall X \in \varpi^n \Lambda' \text{ with } n \geq D, \forall Y \in \varpi^m \Lambda'$$

$$\text{Ad}(\exp X)Y - (Y + \text{ad}(X)Y) \in \varpi^{2n+m-3d} \Lambda.$$

We note that  $\mathcal{A}_N := \mathfrak{m}_1^{\Lambda'} \times \mathbb{C}^1$  is a maximal abelian subgroup of  $\mathcal{H}_N$  stable under  $\theta$ . If we put  $\mathcal{A}_N^+ := \mathcal{A}_N \rtimes \langle \theta \rangle$ , then  $\rho_N^+ = \text{ind}_{\mathcal{A}_N^+}^{\mathcal{H}_N^+} [\text{ind}_{\mathcal{A}_N^+}^{\mathcal{A}_N^+} \chi]$ . Let us denote  $\chi_+$  the extension of  $\chi$  to  $\mathcal{A}_N^+$  such that  $\chi_+(\theta) = 1$ , and take  $\rho_{N,+}$  to be  $\text{ind}_{\mathcal{A}_N^+}^{\mathcal{H}_N^+} \chi_+$  in what follows. Then, writing  $\mathbf{L} := \exp \mathfrak{m}_1^{\Lambda'}$ , we have

$$\mathcal{W}_{N,\phi}(\pi)_\zeta = \text{Hom}_{U^+(F)}(\zeta \bar{\rho}_{N,+}, E_{V,\chi}) = \text{Hom}_{\mathcal{H}_N^+}(\text{ind}_{\mathcal{A}_N^+}^{\mathcal{H}_N^+} \zeta \chi_+, E_{V,\chi})$$

$$\simeq \text{Hom}_{\mathcal{A}_N^+}(\zeta\chi_+, E_{V,\chi}) \simeq (E_{V,\chi}^{\mathbf{L}})_{\zeta}.$$

Here  $(E_{V,\chi}^{\mathbf{L}})_{\zeta}$  denotes the  $\zeta(\theta)$ -eigenspace of  $\theta$  in  $E_{V,\chi}^{\mathbf{L}}$ . In particular (2.1) becomes

$$(5.5) \quad \text{tr}(\theta|\mathcal{W}_{N,\phi}(\pi)) = \sum_{\zeta \in \Xi} \zeta(\theta) \dim(E_{V,\chi}^{\mathbf{L}})_{\zeta}.$$

5.2. SYSTEMS OF  $K$ -TYPES. It follows from (5.3) that  $\mathbf{K}_n := \exp \varpi^n \Lambda'$  is a group which is stable under  $\theta$  for  $n \geq D$ . (5.4) assures that it is normal in  $\mathbf{K}_m$  if  $n \geq m \geq D$ . We shall also need  $\mathbf{K}'_n := \text{Ad}(\phi(\varpi^{-n}))\mathbf{K}_n$ . Introduce the subgroups  $\mathbf{U}_n := U(F) \cap \mathbf{K}_n$ ,  $\bar{\mathbf{P}}_n := \bar{P}(F) \cap \mathbf{K}_n$ ,  $\mathbf{U}'_n := \text{Ad}(\phi(\varpi^{-n}))\mathbf{U}_n$ ,  $\bar{\mathbf{P}}'_n := \text{Ad}(\phi(\varpi^{-n}))\bar{\mathbf{P}}_n$ , where  $\bar{P}$  denotes the parabolic subgroup of  $G$  whose Lie algebra is  $\bar{\mathfrak{p}} := \sum_{i \leq 0} \mathfrak{g}_i$ . We know from [13, I.4, 5] the following:

$$(5.6) \quad \mathbf{K}_n = \bar{\mathbf{P}}_n \mathbf{U}_n, \quad \mathbf{K}'_n = \bar{\mathbf{P}}'_n \mathbf{U}'_n.$$

$$(5.7) \quad \{\bar{\mathbf{P}}_n\}_{n \geq D} \text{ is a system of fundamental neighborhoods of } 1 \text{ in } \bar{P}(F).$$

$$(5.8) \quad \lim_{\substack{\longrightarrow \\ n}} \mathbf{U}'_n = \exp(\Lambda' \cap \mathfrak{g}_1(F))U_{\geq 2}(F), \quad U_{\geq 2} := \exp \sum_{i \geq 2} \mathfrak{g}_i.$$

All of these subgroups are stable under  $\theta$ . We write  $H^+$  for the semidirect product  $H \rtimes \langle \theta \rangle$  where  $H$  is any one of these subgroups.

Also, we have the characters [loc. cit. I.6]

$$\chi_n: \mathbf{K}_n \ni \exp X \mapsto \psi(\varpi^{-2n} B(N, X)) \in \mathbb{C}^\times,$$

and  $\chi'_n := \chi_n \circ \text{Ad}(\phi(\varpi^n)): \mathbf{K}'_n \rightarrow \mathbb{C}^\times$ . These are again  $\theta$ -stable and we extend them trivially on  $\langle \theta \rangle$  to the characters  $\chi_{n,+}$  and  $\chi'_{n,+}$  of  $\mathbf{K}_n^+$  and  $(\mathbf{K}'_n)^+$ , respectively. For  $\zeta \in \Xi$  define

$$E[\zeta\chi_{n,+}] := \{\xi \in E \mid \pi(k)\xi = \zeta\chi_{n,+}(k)\xi, \forall k \in \mathbf{K}_n^+\},$$

$$E[\zeta\chi'_{n,+}] := \{\xi \in E \mid \pi(k)\xi = \zeta\chi'_{n,+}(k)\xi, \forall k \in (\mathbf{K}'_n)^+\},$$

$$I_{n,m}: E[\zeta\chi_{n,+}] \ni \xi \mapsto \frac{1}{\text{meas } \mathbf{K}_m} \int_{\mathbf{K}_m} \overline{\chi_m(k)} \pi(k\phi(\varpi^{m-n}))\xi dk \in E[\zeta\chi_{m,+}],$$

$$I'_{n,m}: E[\zeta\chi'_{n,+}] \ni \xi \mapsto \frac{1}{\text{meas } \mathbf{K}'_m} \int_{\mathbf{K}'_m} \overline{\chi'_m(k)} \pi(k)\xi dk \in E[\zeta\chi'_{m,+}].$$

The commutative diagram [13, I.9 (1)] decomposes as a direct sum of the diagrams:

$$\begin{array}{ccc} E[\zeta\chi_{n,+}] & \xrightarrow{I_{n,m}} & E[\zeta\chi_{m,+}] \\ \pi(\phi(\varpi^{-n})) \downarrow & & \downarrow \pi(\phi(\varpi^{-m})) \\ E[\zeta\chi'_{n,+}] & \xrightarrow{I'_{n,m}} & E[\zeta\chi'_{m,+}] \end{array}$$

## 6. An explicit descent

Recall the function

$$\varphi_n(x) := \begin{cases} \chi_n(x)^{-1} & \text{if } x \in \mathbf{K}_n \\ 0 & \text{otherwise} \end{cases}$$

from [13, I.11]. Here we shall calculate the descent  $\phi_n$  of  $\varphi_n$  to  $G^\theta(F)$ .

6.1. SOME SURJECTIVITY. Write  $\mathfrak{g}(\theta) := (1 - \theta)\mathfrak{g}$  and  $\Lambda'(\theta) := \Lambda' \cap \mathfrak{g}(\theta, F)$ . Put  $\mathbf{K}_n(\theta) := \exp \varpi^n \Lambda'(\theta)$  for  $n \geq D$ .

LEMMA 6.1: *There exists  $n_0 \geq D$  such that the map*

$$\mathbf{K}_n(\theta) \times \mathbf{K}_n^\theta \ni (x, y) \longmapsto xy \in \mathbf{K}_n$$

*is an isomorphism of  $\ell$ -spaces (in the terminology of [3]) for  $n \geq n_0$ .*

*Proof:* Since the derivation of the map at  $(1, 1)$

$$\mathfrak{g}(\theta, F) \times \mathfrak{g}^\theta(F) \ni (X, Y) \longmapsto X + Y \in \mathfrak{g}(F)$$

is surjective, it is submersive. In particular  $\mathbf{K}_D(\theta)\mathbf{K}_D^\theta$  is an open neighborhood of 1 in  $G(F)$ , and we can choose  $n_0 \in \mathbb{Z}_{\geq 0}$  such that this neighborhood contains  $\mathbf{K}_{n_0}$ .

Fix  $n \geq n_0$ . Each  $z \in \mathbf{K}_n$  can be written as  $z = \exp X \exp Y$  with  $X \in \varpi^D \Lambda'(\theta)$ ,  $Y \in \varpi^D \Lambda'^\theta$ . Moreover, if we suppose  $X \in \varpi^r \Lambda'(\theta)$ ,  $Y \in \varpi^r \Lambda'^\theta$ , then (5.3) gives

$$z = \exp\left(X + Y + \frac{[X, Y]}{2} + Z\right),$$

with  $[X, Y]/2 \in \varpi^{2r-3d} \Lambda'$ ,  $Z \in \varpi^{2r+d} \Lambda'$ . This combined with  $z \in \exp \varpi^n \Lambda'$  implies  $X \in \varpi^{r'} \Lambda'(\theta)$ ,  $Y \in \varpi^{r'} \Lambda'^\theta$ , where  $r' := \inf(n, 2r - 3d)$ . Note that  $2r - 3d > r$  for  $r \geq D$ . By repeating this, we conclude that  $\exp X \in \mathbf{K}_n(\theta)$ ,  $\exp Y \in \mathbf{K}_n^\theta$  and the surjectivity is proved. To prove the injectivity, we suppose that  $x, x' \in \mathbf{K}_n(\theta)$  and  $y \in \mathbf{K}_n^\theta$  satisfy  $x' = xy$ . We write  $Y := \log y$ ,  $X := \log x$ . If  $Y \in \varpi^r \Lambda'^\theta$ , then (5.3) gives

$$x' = \exp\left(X + Y + \frac{[X, Y]}{2} + Z\right), \quad \frac{[X, Y]}{2} \in \varpi^{r+n-3d} \Lambda', \quad Z \in \varpi^{n+r+d} \Lambda'.$$

That is,  $Y \in \varpi^{r+n-3d} \Lambda'^\theta$ . Again repeating this, we have  $Y = 0$  and  $x = x'$ . Since the map is smooth and submersive, it is an isomorphism. ■

We have assumed that the order  $\ell$  of  $\theta$  is prime to the residual characteristic  $p$  of  $F$ . Then the polynomial

$$(T+1)^\ell - 1 = \sum_{j=1}^{\ell} \binom{\ell}{j} T^j$$

modulo  $\mathfrak{p}$  cannot have 0 as a root with multiplicity greater than one. In other words,  $\zeta - 1$  is a unit in the integral closure of  $\mathcal{O}$  in an algebraic closure  $\overline{F}$  of  $F$ , for any  $\ell$ -th root of unity  $\zeta$  other than 1. This implies  $(1 - \theta)^{-1}\Lambda'(\theta) = \Lambda'(\theta)$ . Put  $\mathbf{K}_n^{\theta'} := \mathbf{K}_n^\theta \cap G^{\theta'}(F)$ ,  $\mathbf{K}_{n,\theta} := \mathbf{K}_n \cap \Omega_\theta$  (cf. 3.1).

LEMMA 6.2: *There exists  $\nu \geq n_0$  such that*

$$\mathbf{K}_n(\theta) \times \mathbf{K}_n^{\theta'} \ni (g, m) \longmapsto \text{Ad}_\theta(g)m \in \mathbf{K}_{n,\theta}$$

*is an isomorphism of  $\ell$ -spaces for  $n \geq \nu$ .*

*Proof:* Since  $G(F) \times G^{\theta'}(F) \ni (g, m) \longmapsto \text{Ad}_\theta(g)m \in \Omega_\theta$  is submersive, there exists a neighborhood  $\mathcal{U}$  of 1 in  $G(F)$  such that  $\mathcal{U} \cap \Omega_\theta = \text{Ad}_\theta(\mathbf{K}_{n_0})\mathbf{K}_{n_0}^{\theta'}$ . (Note that 1 belongs to the closure of this latter set.) We choose  $\nu \geq n_0$  satisfying  $\mathbf{K}_\nu \subset \mathcal{U}$ . Let  $n \geq \nu$ . Then any  $z \in \mathbf{K}_{n,\theta}$  can be written in the form  $z = \text{Ad}_\theta(g)m$ , ( $g \in \mathbf{K}_{n_0}$ ,  $m \in \mathbf{K}_{n_0}^{\theta'}$ ). Lemma 6.1 allows us to write  $g = x \cdot x^\theta$ , for some  $x \in \mathbf{K}_{n_0}(\theta)$ ,  $x^\theta \in \mathbf{K}_{n_0}^\theta$ . Write  $y := \text{Ad}(x^\theta)m \in \mathbf{K}_{n_0}^{\theta'}$ . Define a sequence  $\{n_k\}_{k \geq 0} \in \mathbb{N}$  starting from  $n_0$  above by the recursion  $n_{k+1} := \inf(n, 2n_k - 3d)$ . If we suppose that  $X := \log x \in \varpi^{n_k}\Lambda'(\theta)$ ,  $Y := \log y \in \varpi^{n_k}\Lambda'^\theta$ , then (5.3) gives

$$\begin{aligned} (6.1) \quad z &= \exp X \exp Y \exp(-\theta(X)) \\ &= \exp\left(X + Y + \frac{[X, Y]}{2} + Z_1\right) \exp(-\theta(X)), \quad \exists Z_1 \in \varpi^{2n_k+d}\Lambda' \\ &= \exp\left((1-\theta)X + Y + \frac{[X, Y]}{2} + Z_1 + \frac{[\theta(X), X + Y + Z_1]}{2} \right. \\ &\quad \left. + \frac{[\theta(X), [X, Y]]}{4} + Z_2\right), \quad \exists Z_2 \in \varpi^{2n_k+d}\Lambda' \\ &= \exp((1-\theta)X + Y + Z_3), \quad Z_3 \in \varpi^{2n_k-3d}\Lambda'. \end{aligned}$$

It follows that  $X \in \varpi^{n_{k+1}}\Lambda'(\theta)$ ,  $Y \in \varpi^{n_{k+1}}\Lambda'^\theta$ , and an induction on  $k$  gives  $x \in \mathbf{K}_n(\theta)$ ,  $y \in \mathbf{K}_n^{\theta'}$ . Conversely, for  $x \in \mathbf{K}_n(\theta)$ ,  $y \in \mathbf{K}_n^{\theta'}$ , (6.1) assures that  $\text{Ad}_\theta(x)y \in \mathbf{K}_{n,\theta}$ . Thus the well-definedness and the surjectivity are proved.

Let us show the injectivity. Suppose  $g, g' \in \mathbf{K}_n(\theta)$  and  $m, m' \in \mathbf{K}_n^{\theta'}$  satisfy  $\text{Ad}_\theta(g)m = \text{Ad}_\theta(g')m'$ . By Lemma 6.1, we write  $g'g^{-1} = xk$  with  $x \in \mathbf{K}_n(\theta)$ ,  $k \in \mathbf{K}_n^\theta$ :

$$\text{Ad}_\theta(x)(\text{Ad}(k)m) = m'.$$

Introduce the sequence  $\{a_k := kn - (3k - 2)d\}_{k \in \mathbb{N}}$ . Suppose that  $X := \log x \in \varpi^{a_k}\Lambda'(\theta)$ ,  $Y := \log \text{Ad}(k)m \in Y' + \varpi^{a_k}\Lambda'^\theta$  with  $Y' := \log m' \in \varpi^n\Lambda'^\theta$ . Then as in (6.1), we see that

$$\exp Y' = \exp((1-\theta)X + Y + Z), \quad Z \in \varpi^{a_k+n-3d}\Lambda' = \varpi^{a_{k+1}}\Lambda',$$

and hence  $X \in \varpi^{a_{k+1}}\Lambda'(\theta)$ ,  $Y \in Y' + \varpi^{a_{k+1}}\Lambda'^\theta$ . Again arguing inductively on  $k$  we see that  $X = 0$  and  $Y = Y'$ . This also gives  $g = g'k$  for some  $k \in \mathbf{K}_n^\theta$ . But then Lemma 6.1 implies  $g = g'$ . Since the map is locally constant and submersive, the above implies that it is an isomorphism. ■

COROLLARY 6.3: *For  $n \geq \nu$ , the map*

$$\mathbf{K}_n \times \mathbf{K}_n^{\theta'} \ni (g, m) \mapsto \mathrm{Ad}_\theta(g)m \in \mathbf{K}_{n,\theta}$$

*is submersive.*

## 6.2. DESCENT FOR $\varphi_n$ .

LEMMA 6.4: *If we set*

$$\alpha_n(g, m) := \begin{cases} \frac{1}{\mathrm{meas} \mathbf{K}_n^\theta} \chi_n(m)^{-1} & \text{if } g \in \mathbf{K}_n, m \in \mathbf{K}_n^{\theta'}, \\ 0 & \text{otherwise,} \end{cases}$$

*then  $\alpha_n \in C_c^\infty(G(F) \times G^{\theta'}(F))$  and we have, for sufficiently large  $n$ ,*

$$\int_{G(F)} \int_{G^\theta(F)} \alpha_n(g, m) \Phi(\mathrm{Ad}_\theta(g)m) dm dg = \int_{G(F)} \varphi_n(g) \Phi(g) dg,$$

*for any  $\Phi \in C_c^\infty(G(F))$ .*

*Proof:* As above, let  $g = \exp X \cdot k \in \mathbf{K}_n$  ( $X \in \varpi^n \Lambda'(\theta)$ ,  $k \in \mathbf{K}_n^\theta$ ),  $m \in \mathbf{K}_n^{\theta'}$ , and write  $Y := \log(\mathrm{Ad}(k)m) \in \varpi^n \Lambda'^\theta$ . (6.1) reads

$$\begin{aligned} \mathrm{Ad}_\theta(g)m = \exp & \left( Y + \frac{[\theta(X), X]}{2} + (1 - \theta)X \right. \\ & \left. + \frac{[(1 + \theta)X, Y]}{2} + \frac{[\theta(X), [X, Y] + 2Z_1]}{4} + Z_1 + Z_2 \right) \end{aligned}$$

for some  $Z_1, Z_2 \in \varpi^{2n+C-d}\Lambda'$ . Since

$$\frac{[\theta(X), [X, Y] + 2Z_1]}{4} + Z_1 + Z_2 \in \varpi^{2n}\Lambda', \quad [(1 + \theta)X, Y] \in \mathfrak{g}(\theta, F),$$

we see that

$$\begin{aligned} \chi_n(\mathrm{Ad}_\theta(g)m) &= \psi(\varpi^{-2n} B(N, \log(\mathrm{Ad}_\theta(g)m))) \\ (6.2) \quad &= \psi(\varpi^{-2n} B(N, Y + \frac{[\theta(X), X]}{2})) \quad (N \in \mathfrak{g}^\theta(F)) \\ &= \chi_n(m) \psi(\varpi^{-2n} \frac{1}{2} B_N(\theta(X), X)) = \chi_n(m). \end{aligned}$$

Note that  $B_N(\Lambda', \Lambda') \subset \mathcal{O}$  gives  $B_N(\theta(X), X)/2 \in \mathfrak{p}^{2n}$ .

Now let  $\alpha_n$  be as in the lemma. For  $\Phi \in C_c^\infty(G(F))$ , we have from Lemma 6.1 that

$$\begin{aligned} & \int_{G(F)} \int_{G^\theta(F)} \alpha_n(g, m) \Phi(\text{Ad}_\theta(g)m) dm dg \\ &= \int_{\mathbf{K}_n(\theta)} \int_{\mathbf{K}_n^\theta} \int_{\mathbf{K}_n^{\theta'}} \alpha_n(xk, m) \Phi(\text{Ad}_\theta(xk)m) dm dk dx \\ &= \int_{\mathbf{K}_n(\theta)} \int_{\mathbf{K}_n^\theta} \int_{\mathbf{K}_n^{\theta'}} \frac{\chi_n(m)^{-1}}{\text{meas } \mathbf{K}_n^\theta} \Phi(\text{Ad}_\theta(x) \text{Ad}(k)m) dm dk dx \end{aligned}$$

writing  $y$  for  $\text{Ad}(k)m$ ,

$$= \int_{\mathbf{K}_n(\theta)} \int_{\mathbf{K}_n^{\theta'}} \chi_n(y)^{-1} \Phi(\text{Ad}_\theta(x)y) dy dx.$$

If we put  $Z = Z^\theta + Z(\theta) := \log(\text{Ad}_\theta(x)y)$ , ( $Z^\theta \in \mathfrak{g}^\theta(F)$ ,  $Z(\theta) \in \mathfrak{g}(\theta, F)$ ),  $X := \log x$ ,  $Y := \log y$ , we have for sufficiently small  $X, Y$  that

$$\left| \frac{\partial(X, Y)}{\partial(Z(\theta), Z^\theta)} \right|_F = \left| \det \begin{pmatrix} (1 - \theta)|_{\mathfrak{g}(\theta)} & \\ & \mathbf{1}_{\mathfrak{g}^\theta} \end{pmatrix} \right|_F^{-1} = |\det(1 - \theta|_{\mathfrak{g}(\theta, F)})|_F^{-1},$$

which is 1 by our hypothesis  $(\ell, p) = 1$  (see the remark preceding Lemma 6.2). Combining this with Lemma 6.2, we deduce

$$\begin{aligned} & \int_{G(F)} \int_{G^\theta(F)} \alpha_n(g, m) \Phi(\text{Ad}_\theta(g)m) dm dg \\ &= \int_{\log \mathbf{K}_{n, \theta}} \chi_n(y)^{-1} \Phi(\exp Z) dZ \stackrel{(6.2)}{=} \int_{\mathbf{K}_{n, \theta}} \chi_n^{-1}(z) \Phi(z) dz \\ &= \int_{G(F)} \varphi_n(g) \Phi(g), \end{aligned}$$

as desired. ■

LEMMA 6.5: The descent  $\phi_n$  of  $\varphi_n$  to  $G^{\theta'}(F)$  is given by

$$\phi_n(m) := \begin{cases} \text{meas } \mathbf{K}_n(\theta) \cdot \chi_n(m)^{-1} & \text{if } m \in \mathbf{K}_n^{\theta'}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof:* We calculate the integral

$$\phi_n(m) = \int_{G(F)} \alpha_n(g, m) dg.$$



Since both sides are zero unless  $m \in \mathbf{K}_n^{\theta'}$ , we assume this. Then Lemma 6.1 gives

$$\begin{aligned} \int_{G(F)} \alpha_n(g, m) dg &= \int_{\mathbf{K}_n(\theta)} \int_{\mathbf{K}_n^\theta} \alpha_n(xk, m) dk dx \\ &= \int_{\mathbf{K}_n(\theta)} \int_{\mathbf{K}_n^\theta} \frac{\chi_n(m)^{-1}}{\text{meas } \mathbf{K}_n^\theta} dk dx \\ &= \text{meas } \mathbf{K}_n(\theta) \cdot \chi_n(m)^{-1}. \quad \blacksquare \end{aligned}$$

## 7. Proof of Theorem 4.1

### 7.1. FROM DEGENERATE WHITTAKER MODELS TO CHARACTER EXPANSIONS.

Let  $\pi \in \Pi(G(F)\theta)$  as in the theorem. We need the following results from [13].

LEMMA 7.1 (Lemma I.13 in [13]): *Suppose that the  $G(F)$ -orbit of  $N \in \mathfrak{g}^\theta(F)$  belongs to  $\mathcal{N}_{\text{Wh}}(\pi)^{\max} = \mathcal{N}_B(\pi)^{\max}$  (cf. [13, I.16]). Let  $X \in \varpi^{[n/2]+b}\Lambda \cap \mathfrak{g}^N(F)$  with  $b > D$ .*

- (i)  $\exp X$  normalizes  $\mathbf{K}_n$  and stabilizes  $\chi_n$ .
- (ii)  $\pi(\exp X)|E[\chi_n] = \text{id}$  for sufficiently large  $n$ .

Let us write  $\mathbf{V}'_n := V(F) \cap \mathbf{K}'_n$ . Note that [13, I.9]

$$\begin{aligned} \chi'_n|_{\bar{\mathbf{P}}'_n} &= \chi_n|_{\bar{\mathbf{P}}_n} = 1, \\ \chi'_n|_{\mathbf{L}} &= \chi'_n|_{\mathbf{U}'_n} = \chi_n|_{\mathbf{U}_n} = 1, \quad \text{for sufficiently large } n, \\ \chi'_n|_{\mathbf{V}'_n} &= \chi|_{\mathbf{V}'_n}, \quad \forall n \geq D. \end{aligned}$$

Using these, it was shown in [*loc. cit.*] that

$$(7.1) \quad I'_{n,m}(\xi) = \frac{1}{\text{meas } \mathbf{V}'_m} \int_{\mathbf{V}'_m} \overline{\chi(v)} \pi(v) \xi dv, \quad \xi \in E[\zeta \chi'_{n,+}], \zeta \in \Xi.$$

If we put

$$E'_{n,\chi} := \bigcup_{m>n} \ker I'_{n,m}, \quad E'_{n,\zeta\chi_+} := \bigcup_{m>n} \ker(I'_{n,m}|E[\zeta\chi'_{n,+}]),$$

then clearly we have  $E'_{n,\chi} = \bigoplus_{\zeta \in \Xi} E'_{n,\zeta\chi_+}$ . The map  $j: E[\chi'_n]/E'_{n,\chi} \rightarrow E_{V,\chi}$ , justified by [3, 2.33] applied to (7.1), also decomposes as a direct sum over  $\zeta \in \Xi$  of

$$j\zeta: E[\zeta\chi'_{n,+}]/E'_{n,\zeta\chi_+} \longrightarrow (E_{V,\chi})_\zeta.$$

We apply this decomposition to [13, Lemma I.14], a consequence of Lemma 7.1, to have

LEMMA 7.2: Suppose  $N \in \mathfrak{g}^\theta(F)$  and  $\mathrm{Ad}(G(F))N \in \mathcal{N}_{\mathrm{Wh}}(\pi)^{\max} = \mathcal{N}_B(\pi)^{\max}$ . Then for sufficiently large  $n$ ,  $j_\zeta$  is injective and its image equals  $(E_{V,\chi}^{\mathbf{L}})_\zeta$ .

Also applying the decomposition  $I_{n,m} = \bigoplus_{\zeta \in \Xi} I_{n,m}|_{E[\zeta\chi_{n,+}]}$  to another consequence [13, I.15] of Lemma 7.1, we have

LEMMA 7.3: Under the same assumption as in the previous lemma, the map  $I_{n,n+1}: E[\zeta\chi_{n,+}] \rightarrow E[\zeta\chi_{n+1,+}]$  is injective for sufficiently large  $n$ .

Noting  $N \in \mathfrak{o} \in \mathcal{N}_{\mathrm{Wh}}^\theta(\pi)^{\max}$  satisfies the assumption of these lemmas, we deduce the following consequences.

PROPOSITION 7.4: Suppose  $N \in \mathfrak{o} \in \mathcal{N}_{\mathrm{Wh}}^\theta(\pi)^{\max}$ . Then we have

$$\mathrm{tr}(\pi(\theta)|E[\chi_n]) = \mathrm{tr}(\theta|\mathcal{W}_{N,\phi}(\pi))$$

for sufficiently large  $n$ . In particular, this is not zero for a suitable choice of  $\phi$ .

*Proof:* If  $n$  is sufficiently large, we have from Lemma 7.2

$$\begin{aligned} \mathrm{tr}(\theta|\mathcal{W}_{N,\phi}(\pi)) &\stackrel{(5.5)}{=} \sum_{\zeta \in \Xi} \zeta(\theta) \dim(E_{V,\chi}^{\mathbf{L}})_\zeta \\ &= \sum_{\zeta \in \Xi} \zeta(\theta) \dim(E[\zeta\chi'_{n,+}]/E'_{n,\zeta\chi_+}) \\ &= \sum_{\zeta \in \Xi} \zeta(\theta) \dim\left(E[\zeta\chi'_{n,+}] / \bigcup_{m>n} \ker(I'_{n,m}|E[\zeta\chi'_{n,+}])\right) \\ &= \sum_{\zeta \in \Xi} \zeta(\theta) \dim E[\zeta\chi_{n,+}] \quad \text{by Lemma 7.3} \\ &= \mathrm{tr}(\pi(\theta)|E[\chi_n]). \quad \blacksquare \end{aligned}$$

COROLLARY 7.5: For any  $\mathfrak{o} \in \mathcal{N}_{\mathrm{Wh}}^\theta(\pi)^{\max}$ , there exists  $\mathfrak{o}' \in \mathcal{N}_{B,\theta}(\pi)$  such that  $\mathfrak{o} \leq \mathfrak{o}'$ .

*Proof:* Thanks to Proposition 7.4 we may assume that  $\mathrm{tr}(\pi(\theta)|E[\chi_n]) \neq 0$  for sufficiently large  $n$  and some  $\phi$ . We take  $n$  sufficiently large so that  $\mathbf{K}_n^{\theta'} \in U_{\pi,\theta}$ . Then Theorem 3.2 combined with Lemma 6.5 gives

$$\mathrm{meas} \mathbf{K}_n \mathrm{tr}(\pi(\theta)|E[\chi_n]) = \Theta_{\pi,\theta}(\varphi_n) = \sum_{\mathfrak{o}' \in \mathcal{N}_{B,\theta}(\pi)} c_{\mathfrak{o}',\theta}(\pi) \widehat{\mu}_{\mathfrak{o}'}(\phi_n \circ \exp).$$

Now we argue as in the proof of [13, I.11] to have

$$\widehat{\mu}_{\mathfrak{o}'}(\phi_n \circ \exp) = \mathrm{meas} \mathbf{K}_n \cdot \mu_{\mathfrak{o}'}(\mathfrak{o}' \cap (\varpi^{-2n}N + \varpi^{-n}(\Lambda'^\theta)^*)),$$

where  $(\Lambda'^\theta)^*$  is the dual lattice of  $\Lambda'^\theta$  in  $\mathfrak{g}^\theta(F)$  with respect to  $\psi \circ B$ . This implies that, since  $\varpi^{2n}\mathfrak{o}' = \mathfrak{o}'$ , for  $\mathrm{tr}(\pi(\theta)|E[\chi_n]) \neq 0$  it is necessary that  $\mathfrak{o}' \cap N + \varpi^n(\Lambda'^\theta)^* \neq \emptyset$ ,  $n \gg 0$  for some  $\mathfrak{o}' \in \mathcal{N}_{B,\theta}(\pi)$ . Since  $\{\varpi^n(\Lambda'^\theta)^*\}_n$  is a system of fundamental neighborhoods of 0, this implies  $N \in \bar{\mathfrak{o}}'$  and the result follows. ■

## 7.2. RELATION BETWEEN THE $K$ -TYPES AND CHARACTER EXPANSIONS.

LEMMA 7.6: For  $N \in \mathfrak{o} \in \mathcal{N}_{B,\theta}(\pi)^{\max}$ , we have

$$\mathrm{tr}(\pi(\theta)|E[\chi_n]) = c_{\mathfrak{o},\theta}(\pi)$$

for sufficiently large  $n$  and any  $\phi$ .

*Proof:* This can be proved in the same manner as [13, I.12]. As in the proof of Corollary 7.5, we have

$$\mathrm{tr}(\pi(\theta)|E[\chi_n]) = \sum_{\mathfrak{o}' \in \mathcal{N}_{B,\theta}(\pi)} c_{\mathfrak{o}',\theta}(\pi) \mu_{\mathfrak{o}'}(\mathfrak{o}' \cap (\varpi^{-2n}N + \varpi^{-n}(\Lambda'^\theta)^*)).$$

Then by the same reasoning as in [*loc. cit.*] the right hand side reduces to the term associated to  $\mathfrak{o}$ :

$$\mathrm{tr}(\pi(\theta)|E[\chi_n]) = c_{\mathfrak{o},\theta}(\pi) \mu_{\mathfrak{o}}(\mathfrak{o} \cap (\varpi^{-2n}N + \varpi^{-n}(\Lambda'^\theta)^*)).$$

What is left is to calculate  $\mu_{\mathfrak{o}}(\mathfrak{X}_n^\theta)$  with  $\mathfrak{X}_n^\theta := \mathfrak{o} \cap (\varpi^{-2n}N + \varpi^{-n}(\Lambda'^\theta)^*)$ . This goes precisely in the same way as [*loc. cit.*] with  $G$  replaced by  $G^\theta$ . The result is that  $\mathfrak{X}_n^\theta$  equals

$$\{\mathrm{Ad}(\exp X \phi(\varpi^n))N \mid X \in \varpi^n(\mathfrak{m}^{\Lambda',\theta})^\theta\},$$

and hence we obtain

$$\begin{aligned} \mu_{\mathfrak{o}}(\mathfrak{X}_n^\theta) &= \mathrm{meas}[\mathbf{K}_n^\theta \phi(\varpi^n) / \mathrm{Ad}(\phi(\varpi^{-n}))(\mathbf{K}_n^\theta \cap G^{\theta,N}(F))] \\ &= |\det[\mathrm{Ad}(\phi(\varpi^{-n}))|_{\mathfrak{g}^\theta(F)/\mathfrak{g}^{\theta,N}(F)}]|_F \cdot \mathrm{meas}(\varpi^n \mathfrak{m}^{\Lambda',\theta}) \\ &= 1. \end{aligned}$$

Here we note, firstly  $\mathrm{meas}(\varpi^n \mathfrak{m}^{\Lambda',\theta}) = |\varpi^{n \dim \mathfrak{m}^\theta}|_F$  by our choice of measures (Remark 3.3), and secondly  $|\det(\mathrm{Ad}(\phi(\varpi^{-n}))|_{(\mathfrak{g}^\theta/\mathfrak{g}^{\theta,N})(F)})|_F$  equals

$$\begin{aligned} &|\det(\mathrm{Ad}(\phi(\varpi^{-n}))|_{\mathfrak{m}^\theta(F)})|_F \\ &= |\varpi^{-n \dim \mathfrak{m}_1^\theta}|_F \cdot \prod_{i \geq 2} |\varpi^{-ni \dim \mathfrak{m}_i^\theta} \varpi^{-n(2-i) \dim \mathfrak{m}_i^\theta}|_F \\ &= |\varpi^{-n \dim \mathfrak{m}^\theta}|_F. \quad \blacksquare \end{aligned}$$

7.3. PROOF OF THE THEOREM. We now prove Theorem 4.1 as the composite of the following two statements.

PROPOSITION 7.7: (i)  $\mathcal{N}_{\text{Wh}}^\theta(\pi)^{\max} \subset \mathcal{N}_{B,\theta}(\pi)^{\max}$ .

(ii) If  $\mathfrak{o} \in \mathcal{N}_{\text{Wh}}^\theta(\pi)^{\max}$ , then for any  $N \in \mathfrak{o}$ ,  $\phi$  and  $\mathfrak{m}$  as in §2.2, we have  $\text{tr}(\theta|\mathcal{W}_{N,\phi}(\pi)) = c_{\mathfrak{o},\theta}(\pi)$ .

*Proof:* (i) For  $\mathfrak{o}_1 \in \mathcal{N}_{\text{Wh}}^\theta(\pi)^{\max}$ , we can take  $\mathfrak{o} \in \mathcal{N}_{B,\theta}(\pi)^{\max}$  with  $\mathfrak{o} \geq \mathfrak{o}_1$  by Corollary 7.5. If we write  $\mathfrak{D}$  and  $\mathfrak{D}_1$  for the elements of  $\mathcal{N}(\mathfrak{g}(F))$  which contain  $\mathfrak{o}$  and  $\mathfrak{o}_1$ , respectively, then  $\mathfrak{D} \geq \mathfrak{D}_1$ . On the other hand, we know from Lemma 7.6 that  $\text{tr}(\pi(\theta)|E[\chi_n]) = c_{\mathfrak{o},\theta}(\pi)$  is not zero for any  $N \in \mathfrak{o}$ ,  $\phi$  and sufficiently large  $n$ . This implies in particular

$$\begin{aligned} 0 \neq \dim E[\chi_n] \text{meas } \mathbf{K}_n &= \Theta_\pi(\varphi_n) = \sum_{\mathfrak{D}' \in \mathcal{N}_B(\pi)} c_{\mathfrak{D}'}(\pi) \widehat{\mu}_{\mathfrak{D}'}(\varphi_n \circ \exp) \\ &= \sum_{\mathfrak{D}' \in \mathcal{N}_B(\pi)} c_{\mathfrak{D}'}(\pi) \mu_{\mathfrak{D}'}(\mathfrak{D}' \cap (\varpi^{-2n}N + \varpi^{-n}(\Lambda')^*)), \end{aligned}$$

by the proof of [13, I.12]. Arguing as in the proof of Corollary 7.5, we can find  $\mathfrak{D}_2 \in \mathcal{N}_B(\pi)$  such that  $N \in \mathfrak{D}_2$ , or equivalently,  $\overline{\mathfrak{D}_1} \subset \overline{\mathfrak{D}} \subset \overline{\mathfrak{D}_2}$ . Since  $\mathfrak{D}_1 \in \mathcal{N}_{\text{Wh}}(\pi)^{\max} = \mathcal{N}_B(\pi)^{\max}$  ([13, I.16]), this forces that  $\mathfrak{D}_1 = \mathfrak{D}_2 = \mathfrak{D}$ . Noting that  $\mathfrak{D} \cap \mathfrak{g}^\theta(F)$  is a finite union of nilpotent  $G^\theta(F)$ -orbits of the same dimension, we conclude from  $\bar{\mathfrak{o}}_1 \subset \bar{\mathfrak{o}}$  that  $\mathfrak{o}_1 = \mathfrak{o} \in \mathcal{N}_{B,\theta}(\pi)^{\max}$ .

(ii) For arbitrary  $N \in \mathfrak{o}$  and  $\phi \in X_*(G^\theta)_F$  satisfying the conditions, Proposition 7.4 gives

$$\text{tr}(\theta|\mathcal{W}_{N,\phi}(\pi)) = \text{tr}(\pi(\theta)|E[\chi_n]), \quad n \gg 0.$$

Thanks to (i), we can apply Lemma 7.6 to have  $\text{tr}(\pi(\theta)|E[\chi_n]) = c_{\mathfrak{o},\theta}(\pi)$ . ■

THEOREM 7.8:  $\mathcal{N}_{\text{Wh}}^\theta(\pi)^{\max} = \mathcal{N}_B^\theta(\pi)^{\max}$ .

*Proof:*  $\mathcal{N}_{\text{Wh}}^\theta(\pi)^{\max} \subset \mathcal{N}_B^\theta(\pi)^{\max}$  is clear from the previous proposition. Conversely, if  $\mathfrak{o} \in \mathcal{N}_B^\theta(\pi)^{\max} \subset \mathcal{N}_{B,\theta}(\pi)^{\max}$ , then the  $G(F)$ -orbit  $\mathfrak{D}$  containing  $\mathfrak{o}$  belongs to  $\mathcal{N}_{\text{Wh}}(\pi)^{\max}$  by [13], and  $\text{tr}(\theta|\mathcal{W}_{N,\phi}(\pi))$  is finite. Since  $N \in \mathfrak{o}$  satisfies the assumption of Lemmas 7.1, 7.2 and 7.3, the proof of Proposition 7.4 also applies in this case to give

$$\text{tr}(\theta|\mathcal{W}_{N,\phi}(\pi)) = \text{tr}(\pi(\theta)|E[\chi_n]) \stackrel{\text{Lemma 7.6}}{=} c_{\mathfrak{o},\theta}(\pi) \neq 0.$$

Thus we have  $\mathfrak{o} \in \mathcal{N}_{\text{Wh}}^\theta(\pi)^{\max}$ . ■

## 8. Twisted endoscopy implies the generic packet conjecture

8.1. TWISTED ENDOSCOPY PROBLEMS TO BE CONSIDERED. The fundamentals of the general theory of twisted endoscopy are exploited in [12]. Here we treat some very special but important examples of the theory introduced by Arthur [2, §9]. As for notation about endoscopy, we refer the reader to Appendix A.

Let  $F$  be a  $p$ -adic field of odd residual characteristic. Recall that a twisted endoscopy problem is attached to a triple  $(G, \theta, \mathbf{a})$ , where  $G$  is a connected reductive group over  $F$ ,  $\theta$  is a quasi-semisimple automorphism of  $G$  and  $\mathbf{a}$  is a class in  $H^1(W_F, Z(\widehat{G}))$ .  $\mathbf{a}$  can be considered as the equivalence class of Langlands parameters attached to a character  $\omega$  of  $G(F)$ . Then the theory is intended to study those irreducible admissible representations or “packets”  $\Pi$  of  $G(F)$  satisfying  $\Pi \circ \theta \simeq \omega \otimes \Pi$ .

Let  $E$  be a quadratic extension of  $F$  or  $F$  itself. Write  $\sigma$  for the generator of the Galois group  $\Gamma_{E/F}$  of this extension. We shall be concerned with the following triple  $(L, \theta, \mathbf{1})$ .  $L := \text{Res}_{E/F} GL(n)$ .  $\theta := \theta_n \circ \tilde{\sigma}$  where  $\theta_n$  is the automorphism

$$\theta_n(g) := \text{Ad}(I_n)({}^t g^{-1}), \quad I_n := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \ddots & & \\ (-1)^{n-1} & & & \end{pmatrix},$$

and  $\tilde{\sigma}$  denotes the  $F$ -automorphism of  $L$  attached to  $\sigma$  by the  $F$ -structure of  $GL(n)$ . The sets of endoscopic data which we shall consider are given as follows [2, §9] (see also Appendix A).

CASE (A):  $E \neq F$ . In this case, we have  ${}^L L = GL(n, \mathbb{C})^2 \rtimes_{\rho_L} W_F$  with

$$\rho_L(w)(g_1, g_2) = \begin{cases} (g_1, g_2) & \text{if } w \in W_E, \\ (g_2, g_1) & \text{otherwise.} \end{cases}$$

The set of endoscopic data is  $(G, {}^L G, s, \xi)$ , where  $G$  is the quasisplit unitary group  $U_{E/F}(n)$  in  $n$  variables attached to  $E/F$ ,  $s := (\mathbf{1}_n, \mathbf{1}_n) \in \widehat{G}$  and  $\xi: {}^L G \hookrightarrow {}^L L$  is given by

$$\xi(g \rtimes_{\rho_G} w) = (g, \theta_n(g)) \rtimes_{\rho_L} w, \quad g \in \widehat{G} = GL(n, \mathbb{C}), \quad w \in W_F.$$

CASE (B):  $E = F$ ,  $n = 2m$  is even. In this case  ${}^L L = GL(n, \mathbb{C}) \times W_F$ . The set of data is  $(G, {}^L G, s, \xi)$  where  $G = SO(2m+1)$  is the split orthogonal group in  $2m+1$  variables,  $s := \mathbf{1}_{2m}$  and

$$\xi: {}^L G \ni g \times w \mapsto g \times w \in {}^L L.$$

Here  $\widehat{G} = Sp(m, \mathbb{C})$  is realized with respect to  $I_{2m}$ .

CASE (C):  $E = F$ ,  $n = 2m + 1$  is odd. The set of data  $(G, {}^L G, s, \xi)$  is given by  $G = Sp(n)$ ,  $s = \mathbf{1}_{2m+1}$ ,

$$\xi: {}^L G \ni g \times w \mapsto g \times w \in {}^L L.$$

Here  $\widehat{G} = SO(2m + 1, \mathbb{C})$  is realized with respect to  $I_{2m+1}$ .

CASE (D):  $E = F$ ,  $n = 2m$  is even. Let  $K$  be a quadratic extension of  $F$  or  $F$  itself. The set of data is  $(G, {}^L G, s, \xi)$ . Here  $G$  is the quasisplit orthogonal group which is isomorphic to the split group  $SO(2m)$  over  $K$ , and on which the non-trivial element  $\tau$  of  $\text{Gal}(K/F)$  (if it exists) acts by the unique non-trivial element of  $\text{Int}(O(2m))/\text{Int}(SO(2m))$ .  $s := \text{diag}(\mathbf{1}_m, -\mathbf{1}_m)$  and  $\xi$  is given by

$$\xi(g \rtimes_{\rho_G} w) = \begin{cases} g \times w & \text{if } w \in W_K, \\ g \begin{pmatrix} & I_m \\ {}^t I_m & \end{pmatrix} \times w & \text{otherwise.} \end{cases}$$

Here  $\widehat{G} = SO(2m, \mathbb{C})$  is realized with respect to  $(I_m \quad I_m)$ .

**8.2. WORKING HYPOTHESES.** We now make some assumptions on the harmonic analytic aspects of the twisted endoscopy for  $(L, \theta, 1)$  and  $(G, {}^L G, s, \xi)$  introduced above. For any  $\delta \in L$ , we write  $L^{\delta, \theta}$  for the group of points in  $L$  fixed under  $\text{Ad}(\delta) \circ \theta$  and  $L_{\delta, \theta}$  for its identity component.  $\delta \in L$  is  **$\theta$ -semisimple** if  $\text{Ad}(\delta) \circ \theta$  induces a semisimple automorphism of the Lie algebra  $\text{Lie } L_{\text{der}}$  of the derived group of  $L$ . A  $\theta$ -semisimple  $\delta \in L$  is  **$\theta$ -regular** if  $L_{\delta, \theta}$  is a torus, and **strongly  $\theta$ -regular** if  $L^{\delta, \theta}$  is abelian. We write  $L_{\theta, \text{sr}}(F)$  for the set of strongly  $\theta$ -regular elements in  $L(F)$ . At each  $\delta \in L_{\theta, \text{sr}}(F)$  we define the  **$\theta$ -orbital integral** by

$$O_{\delta, \theta}(f) := \int_{L_{\delta, \theta}(F) \backslash L(F)} f(g^{-1} \delta \theta(g)) \frac{dg}{dt}.$$

Two strongly  $\theta$ -regular  $\delta, \delta' \in L(F)$  are **stably  $\theta$ -conjugate** if they are  $\theta$ -conjugate in  $L(\bar{F})$ . We define the **stable  $\theta$ -orbital integral** at  $\delta \in L_{\theta, \text{sr}}(F)$  by

$$SO_{\delta, \theta}(f) := \sum_{\substack{\delta' \text{ stably } \theta\text{-conj. to } \delta \\ \text{mod. } \theta\text{-conj.}}} O_{\delta', \theta}(f).$$

In [12, Ch. 3], Kottwitz and Shelstad constructed the (strongly regular) **norm map**, which we denote by  $N_{L/G}$ , from the set of stable  $\theta$ -conjugacy classes in  $L_{\theta, \text{sr}}(F)$  to that of strongly regular stable conjugacy classes in  $G(F)$ . Also they defined a function  $\Delta_{L/G}(\gamma, \delta)$  on  $G_{\text{sr}}(F) \times L_{\theta, \text{sr}}(F)$  called the **transfer factor**.

Of course their construction applies to the most general setting. In our case, we know that

$$(8.1) \quad \Delta_{L/G}(\gamma, \delta) = \begin{cases} 1 & \text{if } \gamma \in N_{L/G}(\delta), \\ 0 & \text{otherwise.} \end{cases}$$

To define the endoscopic lifting, we need the following conjecture.

CONJECTURE 8.1 (Transfer conjecture): *For  $f \in C_c^\infty(L(F))$ , there exists  $f^G \in C_c^\infty(G(F))$  such that*

$$SO_\gamma(f^G) = \sum_{\delta} \Delta_{L/G}(\gamma, \delta) O_{\delta, \theta}(f).$$

Here  $\delta$  runs over the  $\theta$ -conjugacy classes whose norm contains  $\gamma$ .

As opposed to the ordinary (i.e.,  $\theta = \text{id}$ ) case, we do not have the precise notion of stable distributions in the twisted case. But we assume this in the following. We also have to postulate the existence of discrete  $L$ -packets. An irreducible admissible representation  $\pi$  of  $G(F)$  is **square integrable** if it appears discretely in Harish-Chandra's Plancherel formula for  $G(F)$ . The set of isomorphism classes of such representations is denoted by  $\Pi_{\text{disc}}(G(F))$ .

CONJECTURE 8.2: (1)  $\Pi_{\text{disc}}(G(F))$  is partitioned into a disjoint union of finite sets of representations  $\Pi_\varphi$  called (discrete)  $L$ -packets:

$$\Pi_{\text{disc}}(G(F)) = \coprod_{\varphi \in \Phi_{\text{disc}}(G(F))} \Pi_\varphi.$$

(2) There exists a function  $\delta(1, \bullet): \Pi_\varphi \rightarrow \mathbb{C}^\times$  such that

$$\Theta_\varphi := \sum_{\pi \in \Pi_\varphi} \delta(1, \pi) \Theta_\pi$$

is a stable distribution.

An irreducible admissible representation of  $G$  is tempered if it contributes non-trivially to the Plancherel formula. Let  $P = MU$  be a  $F$ -parabolic subgroup of  $G$  and  $\tau \in \Pi_{\text{disc}}(M(F))$ . Then the induced representation  $\text{ind}_{P(F)}^{G(F)}[\tau \otimes \mathbf{1}_{U(F)}]$  is a direct sum of irreducible tempered representations of  $G(F)$ :

$$\text{ind}_{P(F)}^{G(F)}[\tau \otimes \mathbf{1}_{U(F)}] \simeq \bigoplus_{i=1}^{\ell_\tau} \pi_i(\tau).$$

Moreover, any irreducible tempered representation of  $G(F)$  is obtained in this way for some  $(M, \tau)$  unique up to  $G(F)$ -conjugation. Regarding this, we define a **tempered  $L$ -packet** by

$$\Pi_\varphi := \prod_{\tau \in \Pi_\varphi^M} \{\pi_i(\tau) \mid 1 \leq i \leq \ell_\tau\},$$

where  $\Pi_\varphi^M$  is a discrete  $L$ -packet of  $M$ . By putting  $\delta(1, \pi_i(\tau)) := \delta(1, \tau)$ , Conjecture 8.2 with  $\Pi_{\text{disc}}(G(F))$  replaced by the set  $\Pi_{\text{temp}}(G(F))$  of the isomorphism classes of irreducible tempered representations of  $G(F)$  follows.

Finally, we say that  $\pi \in \Pi(L(F)\theta)$  is  **$\theta$ -discrete** if it is tempered and is not induced from a  $\theta$ -stable tempered representation of a proper Levi subgroup. We write  $\Pi_{\text{disc}}(L(F)\theta)$  for the subset of  $\theta$ -discrete elements in  $\Pi(L(F)\theta)$ . Note that each  $\pi \in \Pi_{\text{disc}}(L(F)\theta)$  is generic [20]. Now we can define the twisted endoscopic lifting which we need.

**CONJECTURE 8.3:** *There should be a bijection  $\xi$  from the set  $\Phi_{\text{disc}}(G(F))$  of discrete  $L$ -packets of  $G(F)$  to  $\Pi_{\text{disc}}(L(F)\theta)$ , which should be characterized by*

$$\Theta_{\xi(\Pi), \theta}(f) = c \cdot \Theta_\Pi(f^G),$$

for any  $f \in C_c^\infty(L(F))$  and  $f^G \in C_c^\infty(G(F))$  as in Conjecture 8.1. Here  $c$  is some non-zero constant.

Although we can be more explicit about the constant  $c$  if we adopt the Whittaker normalization of the transfer factor [12, 5.3], it is not necessary for our purpose.

**8.3. TE IMPLIES GPC.** Now we prove the following.

**THEOREM 8.4:** *Assume Conjecture 8.3. Then the generic packet conjecture holds for  $G$ .*

Write  $\mathfrak{l} := \text{Lie } L$ . For  $h \in C_c^\infty(\mathfrak{l}(F))$  and  $t \in F^\times$ , we put  $h_t(X) := h(t^{-1}X)$ , ( $X \in \mathfrak{l}(F)$ ). We assume that the support of  $f \in C_c^\infty(L(F))$  is sufficiently small so that there exists a neighborhood  $\mathcal{V}$  of 0 in  $\mathfrak{l}(F)$ , on which the exponential map is defined and injective, satisfying  $\text{supp } f \subset \exp(\mathcal{V})$ . Then we can consider  $f \circ \exp \in C_c^\infty(\mathfrak{l}(F))$ . Taking  $t$  sufficiently small, we may define  $f_t \in C_c^\infty(L(F))$  by  $f_t \circ \exp := (f \circ \exp)_t$ . Further, we might take  $f$  and  $\mathcal{V}$  so that the transferred function  $f^G$  satisfies the same condition. We define  $f_t^G$  in the same fashion. As in [16, Lem. 9.7], one can deduce from (8.1) the following:



LEMMA 8.5: Let  $f \in C_c^\infty(L(F))$  and  $f^G \in C_c^\infty(G(F))$  be as in Conjecture 8.1. Suppose that  $\text{supp } f$  is so small that we can define  $f_t$  and  $f_t^G$  for sufficiently small  $t$ . Then we have

$$SO_\gamma(f_{t^2}^G) = \sum_{\delta} \Delta_{L/G}(\gamma, \delta) O_{\delta, \theta}(f_{t^2}),$$

for  $t \in F^\times$  small enough.

Let us prove the theorem. Since  $\text{ind}_{P(F)}^{G(F)}[\tau \otimes \mathbf{1}_{U(F)}]$  is generic if  $\tau$  is so, we are reduced to the case of a discrete  $L$ -packet  $\Pi$ . Then by Conjecture 8.3, we have

$$\Theta_{\xi(\Pi), \theta}(f) = \sum_{\pi \in \Pi} \delta(1, \pi) \Theta_\pi(f^G).$$

Suppose that  $\text{supp } f$  is sufficiently small. Then applying the asymptotic expansions (3.1) and (4.1) to the left and right hand sides respectively, we have

$$\sum_{\mathfrak{D} \in \mathcal{N}(\mathfrak{l}^\theta(F))} c_{\mathfrak{D}, \theta}(\xi(\Pi)) \widehat{\mu}_{\mathfrak{D}}(f^\theta \circ \exp) = \sum_{\mathfrak{o} \in \mathcal{N}(\mathfrak{g}(F))} \sum_{\pi \in \Pi} \delta(1, \pi) c_{\mathfrak{o}}(\pi) \widehat{\mu}_{\mathfrak{o}}(f^G \circ \exp).$$

Here  $f^\theta \in C_c^\infty(G^\theta(F))$  is the descent of  $f$ .

Let  $\mathfrak{o} \in \mathcal{N}(\mathfrak{g}(F))$  and  $N \in \mathfrak{o}$ . We say that  $\mathfrak{o}$  is  **$r$ -regular** if the variety  $\mathfrak{B}_N$  of Borel subalgebras of  $\mathfrak{g}$  containing  $N$  is  $r$ -dimensional. It is a result of Harish-Chandra that

$$\widehat{\mu}_{\mathfrak{o}}(f_{t^2}^G \circ \exp) = |t|_F^{r - \ell(G)} \widehat{\mu}_{\mathfrak{o}}(f^G \circ \exp)$$

for an  $r$ -regular  $\mathfrak{o}$  [9, Lem. 22]. Here  $\ell(G)$  denotes the dimension of the flag variety of  $G$ . The same is true for  $\mathfrak{l}^\theta$ . Now recall that  $\xi(\Pi)$  is generic. That is, for any 0-regular nilpotent  $N$  and  $\phi$  as in §2.2, we have  $\mathcal{W}_{N, \phi}(\xi(\Pi)) \neq 0$ . It follows from the uniqueness of the Whittaker model that  $\text{tr}(\xi(\Pi)(\theta) | \mathcal{W}_{N, \phi}(\xi(\Pi))) = 1$ , and hence  $c_{\mathfrak{D}, \theta}(\xi(\Pi)) = 1$  for the regular nilpotent orbit  $\mathfrak{D}$ . Thus in the equality

$$\sum_{\mathfrak{D} \in \mathcal{N}(\mathfrak{l}^\theta(F))} c_{\mathfrak{D}, \theta}(\xi(\Pi)) \widehat{\mu}_{\mathfrak{D}}(f_{t^2}^\theta \circ \exp) = \sum_{\mathfrak{o} \in \mathcal{N}(\mathfrak{g}(F))} \sum_{\pi \in \Pi} \delta(1, \pi) c_{\mathfrak{o}}(\pi) \widehat{\mu}_{\mathfrak{o}}(f_{t^2}^G \circ \exp),$$

the term of order  $-\ell(L^\theta) = -\ell(G)$  in  $|t|_F$  on the left hand side is not zero. Thus  $c_{\mathfrak{o}}(\pi)$  is not zero for at least one regular  $\mathfrak{o}$ . This combined with [13, I.16, 17] implies the genericity of  $\Pi$ .

## Appendix A. Twisted endoscopic data for $GL(n)$

Here we classify the isomorphism classes of the sets of elliptic endoscopic data for the triple  $(L, \theta, 1)$  in §8.1 over  $F$ , a local or global field of characteristic zero.

Fixing an algebraic closure  $\bar{F}$  of  $F$ , we write  $\Gamma_F$  and  $W_F$  for the absolute Galois group and the Weil group of  $\bar{F}/F$ . If  $F$  is global, we write  $\mathbb{A}_F$  for the adèle ring of  $F$ . For convenience we take the  $\theta$ -invariant  $F$ -splitting  $\mathbf{spl}_L := (\mathbf{B}, \mathbf{T}, \{X_\alpha\})$  of  $L$  coming from the standard splitting  $\mathbf{spl}_n = (\mathbf{B}_n, \mathbf{T}_n, \{X_{\alpha_i}\}_{i=1}^n)$  of  $GL(n)$ . Recall that the  $L$ -group  ${}^L L = \hat{L} \rtimes_{\rho_L} W_F$  is given by

$$\hat{L} = GL(n, \mathbb{C})^{[E:F]}, \quad \rho_L(w)(g, h) = \begin{cases} (g, h) & \text{if } w \in W_E, \\ (h, g) & \text{otherwise.} \end{cases}$$

A.1. DEFINITIONS. We return to a general  $(G, \theta, \mathbf{a})$  in this subsection. We take a splitting  $\mathbf{spl}_{\hat{G}} := (\mathcal{B}, \mathcal{T}, \{\mathcal{X}_\alpha\})$  of  $\hat{G}$  fixed under the  $\Gamma_F$ -action  $\rho_G$ . The dual of the inner class of  $\theta$  is an automorphism of the based root datum of  $\hat{G}$ . This lifts to an automorphism  $\hat{\theta}$  of  $\hat{G}$  which preserves  $\mathbf{spl}_{\hat{G}}$ . In the case  $G = L$ , we take  $\mathbf{spl}_L$  to be the standard one for  $GL(n, \mathbb{C})^{[E:F]}$ . Then  $\hat{\theta}$  becomes

$$\begin{cases} \hat{\theta}(g, h) = (\theta_n(h), \theta_n(g)) & \text{if } [E:F] = 2, \\ \hat{\theta}(g) = \theta_n(g) & \text{otherwise.} \end{cases}$$

Recall from [TE] that a quadruple  $(H, \mathcal{H}, s, \xi)$  is a **set of endoscopic data** for  $(G, \theta, \mathbf{a})$  if

- $H$  is a quasisplit connected reductive  $F$ -group. We fix an  $L$ -group datum  $(\hat{H}, \rho_H, \eta_H)$ .
- $\mathcal{H}$  is a split extension

$$1 \longrightarrow \hat{H} \longrightarrow \mathcal{H} \xrightarrow{\pi} W_F \longrightarrow 1.$$

Thus we have an injective homomorphism  $\iota: W_F \hookrightarrow \mathcal{H}$  satisfying  $\pi \circ \iota = \text{id}_{W_F}$ . We impose that the inner class of  $\text{Ad}(\iota(w))|_{\hat{H}}$  coincides with that of  $\rho_H(w)$  for any  $w \in W_F$ .

- $s$  is a  $\hat{\theta}$ -semisimple element in  $\hat{G}$  (cf. §8.2).
- $\xi: \mathcal{H} \hookrightarrow {}^L G$  is an  $L$ -embedding satisfying

$$(A.1) \quad \text{Ad}(s) \circ \hat{\theta} \circ \xi = a' \cdot \xi, \quad \exists a' \in \mathbf{a},$$

$$(A.2) \quad \xi(\hat{H}) = \hat{G}_{s, \hat{\theta}}.$$

A set of endoscopic data  $(H, \mathcal{H}, s, \xi)$  is **elliptic** if  $\xi(Z(\hat{H})^{\Gamma_F})^0 \subset Z(\hat{G})$ . Two elliptic sets of data  $(H, \mathcal{H}, s, \xi)$  and  $(H', \mathcal{H}', s', \xi')$  are **isomorphic** if there exists  $g \in \hat{G}$  such that

$$(A.3) \quad \xi'(\mathcal{H}') = \text{Ad}(g)\xi(\mathcal{H}),$$

$$(A.4) \quad s' \in \text{Ad}(g)s \cdot Z(\hat{G}).$$

A.2.  $\widehat{\theta}$ -SEMISIMPLE CLASSES IN  $\widehat{L}$ . We name the cases we consider as follows.

- (A)  $E$  is a quadratic extension of  $F$ .
- (B)  $E = F$  and  $n = 2n'$  for some  $n' \in \mathbb{N}$ .
- (C)  $E = F$  and  $n = 2n' + 1$  for some  $n' \in \mathbb{N}$ .

To classify the endoscopic data, we begin with the classification of  $s$  and  $\widehat{H}$ . Recall that our  $\widehat{\theta}$  preserves  $\mathcal{T}$ . Set  $\mathcal{T}(\widehat{\theta}) := (1 - \widehat{\theta})\mathcal{T}$  and  $\mathcal{T}_{\widehat{\theta}} := \mathcal{T}/\mathcal{T}(\widehat{\theta})$ . Introduce the **absolute norm map**

$$N_{\widehat{\theta}}: \mathcal{T} \ni t \longmapsto t\widehat{\theta}(t) \in \mathcal{T}.$$

The following lemma follows from a simple calculation.

LEMMA A.1:  $\ker N_{\widehat{\theta}} = \mathcal{T}(\widehat{\theta})$ . Hence we can identify  $\mathcal{T}_{\widehat{\theta}}$  with  $\text{Im } N_{\widehat{\theta}}$ .

The set of  $\widehat{\theta}$ -semisimple  $s \in \widehat{L}$  up to isomorphisms is simply that of  $\widehat{\theta}$ -semisimple  $\widehat{\theta}$ -conjugacy classes in  $\widehat{L}$  modulo  $Z(\widehat{L})$ . Thanks to [12, Lem. 3.2.A], this set is in bijection with  $Z(\widehat{L})_{\widehat{\theta}} \backslash \mathcal{T}_{\widehat{\theta}} / \Omega^{\theta}$ , where  $Z(\widehat{L})_{\widehat{\theta}} := Z(\widehat{L}) / Z(\widehat{L}) \cap \mathcal{T}(\widehat{\theta})$ ,  $\Omega$  is the Weyl group of  $\mathcal{T}$  in  $\widehat{L}$  and  $\Omega^{\theta}$  is its  $\widehat{\theta}$ -fixed part. Using Lemma A.1, we identify  $\mathcal{T}_{\widehat{\theta}}$  with

$$\begin{cases} \{(\text{diag}(t_1, \dots, t_n), \text{diag}(t_n^{-1}, \dots, t_1^{-1})) \mid t_i \in \mathbb{C}^{\times}\} & \text{in case (A),} \\ \{\text{diag}(t_1, \dots, t_{n'}, t_{n'}^{-1}, \dots, t_1^{-1}) \mid t_i \in \mathbb{C}^{\times}\} & \text{in case (B),} \\ \{\text{diag}(t_1, \dots, t_{n'}, 1, t_{n'}^{-1}, \dots, t_1^{-1}) \mid t_i \in \mathbb{C}^{\times}\} & \text{in case (C).} \end{cases}$$

Write  $\widehat{L}_{\widehat{\theta}}$  for the identity component of  $\widehat{L}^{\widehat{\theta}}$ . Since  $\widehat{\theta}$  preserves  $\mathbf{spl}_{\widehat{L}}$ ,  $\Omega^{\theta}$  equals the Weyl group of  $\mathcal{T}_{\widehat{\theta}} = \text{Im } N_{\widehat{\theta}}$  in  $\widehat{L}_{\widehat{\theta}}$ . We have

$$\widehat{L}_{\widehat{\theta}} = \begin{cases} \{(g, \theta_n(g)) \mid g \in GL(n, \mathbb{C})\} & \text{in case (A),} \\ Sp(n', \mathbb{C}) & \text{in case (B),} \\ SO(2n' + 1, \mathbb{C}) & \text{in case (C).} \end{cases}$$

The action of  $\Omega(\widehat{L}_{\widehat{\theta}}, \mathcal{T}_{\widehat{\theta}})$  on  $\mathcal{T}_{\widehat{\theta}}$  is obvious. Noting the identification

$$Z(\widehat{L})_{\widehat{\theta}} = N_{\widehat{\theta}}(Z(\widehat{L})) = \begin{cases} \{(z\mathbf{1}_n, z^{-1}\mathbf{1}_n) \mid z \in \mathbb{C}^{\times}\} & \text{if } [E : F] = 2, \\ \{1\} & \text{otherwise,} \end{cases}$$

we obtain the following. For  $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{N}^r$  and  $\underline{\lambda} = (\lambda_1, \dots, \lambda_r) \in (\mathbb{C}^{\times})^r$ , we write  $d_{\mathbf{m}}(\underline{\lambda}) = \text{diag}(\lambda_1\mathbf{1}_{m_1}, \dots, \lambda_r\mathbf{1}_{m_r})$ . Similarly set  $d_{\mathbf{m}}(g) := \text{diag}(g_1, \dots, g_r)$  for  $\underline{g} = (g_i)_{i=1}^r \in \prod_{i=1}^r GL(m_i)$ . Write  $|\mathbf{m}| = \sum_{i=1}^r m_i$  for the length of  $\mathbf{m}$ , and  $r(\mathbf{m}) := r$ .

LEMMA A.2: The  $\widehat{\theta}$ -semisimple elements  $s$  up to isomorphisms are given as follows.

(A) There exists a partition  $\mathbf{m}$  of  $n$  and  $\underline{\lambda} \in (\mathbb{C}^\times)^{r(\mathbf{m})}$  with  $\lambda_i \neq \lambda_j$  for  $i \neq j$ , such that  $N_{\hat{\theta}}(s) = (d_{\mathbf{m}}(\underline{\lambda}), \theta_n(d_{\mathbf{m}}(\underline{\lambda})))$  and  $s = (\mathbf{1}_n, \theta_n(d_{\mathbf{m}}(\underline{\lambda})))$ .

(B), (C) There exists  $\mathbf{m}$  with  $|\mathbf{m}| \leq n'$  and  $\underline{\lambda} \in (\mathbb{C}^\times)^{r(\mathbf{m})}$  with  $\lambda_i \neq \lambda_j \neq \pm 1$  for  $i \neq j$ , such that

$$N_{\hat{\theta}}(s) = \begin{cases} \text{diag}(d_{\mathbf{m}}(\underline{\lambda}), -\mathbf{1}_{m'}, \mathbf{1}_{2m}, -\mathbf{1}_{m'}, \theta_{|\mathbf{m}|}(d_{\mathbf{m}}(\underline{\lambda}))) & \text{in case (B),} \\ \text{diag}(d_{\mathbf{m}}(\underline{\lambda}), -\mathbf{1}_{m'}, \mathbf{1}_{2m+1}, -\mathbf{1}_{m'}, \theta_{|\mathbf{m}|}(d_{\mathbf{m}}(\underline{\lambda}))) & \text{in case (C),} \end{cases}$$

$$s = \begin{cases} \text{diag}(\mathbf{1}_{|\mathbf{m}|}, \mathbf{1}_{m'}, \mathbf{1}_{2m}, -\mathbf{1}_{m'}, \theta_{|\mathbf{m}|}(d_{\mathbf{m}}(\underline{\lambda}))) & \text{in case (B),} \\ \text{diag}(\mathbf{1}_{|\mathbf{m}|}, \mathbf{1}_{m'}, \mathbf{1}_{2m+1}, -\mathbf{1}_{m'}, \theta_{|\mathbf{m}|}(d_{\mathbf{m}}(\underline{\lambda}))) & \text{in case (C).} \end{cases}$$

A.3. TWISTED CENTRALIZERS. Next we classify  $\hat{H} = \hat{L}_{s, \hat{\theta}}$ . Our strategy is standard: First calculate the (connected) centralizer  $\hat{L}_{N_{\hat{\theta}}(s)}$ , then determine the fixed part  $\hat{L}^{s, \hat{\theta}}$  of the involution  $\text{Ad}(s) \circ \hat{\theta}$  on  $\hat{L}_{N_{\hat{\theta}}(s)}$ .

From the above lemma  $\hat{L}_{N_{\hat{\theta}}(s)}$  is given by

$$\{(d_{\mathbf{m}}(\underline{g}), \theta_n(d_{\mathbf{m}}(\underline{g}')) \mid \underline{g}, \underline{g}' \in \prod_{i=1}^r GL(m_i, \mathbb{C})\}$$

in case (A), and

$$\left\{ \text{diag}(d_{\mathbf{m}}(\underline{g}), \begin{pmatrix} a & b \\ & g \\ c & d \end{pmatrix}, \theta_{|\mathbf{m}|}(d_{\mathbf{m}}(\underline{g}')) \mid \begin{array}{l} \underline{g}, \underline{g}' \in \prod_{i=1}^r GL(m_i, \mathbb{C}) \\ \underline{g}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2m', \mathbb{C}) \\ g \in GL(n_0, \mathbb{C}) \end{array} \right\}$$

in cases (B) and (C). Here  $n_0 = 2m$  in case (B) and  $2m + 1$  in case (C). Then one can easily see that  $\text{Ad}(s) \circ \hat{\theta}$  restricted to  $\hat{L}_{N_{\hat{\theta}}(s)}$  acts as  $(d_{\mathbf{m}}(\underline{g}), d_{\mathbf{m}}(\underline{g}')) \mapsto (\theta_{|\mathbf{m}|}(d_{\mathbf{m}}(\underline{g})), \theta_{|\mathbf{m}|}(d_{\mathbf{m}}(\underline{g}'))$  on the  $\prod_{i=1}^r GL(m_i, \mathbb{C})$ -component,

$$g' \mapsto \begin{cases} \text{Ad}\left(\begin{pmatrix} \mathbf{0}_{m'} & I_{m'} \\ {}^t I_{m'} & \mathbf{0}_{m'} \end{pmatrix}\right)({}^t g^{-1}) & \text{in case (B),} \\ \text{Ad}\left(\begin{pmatrix} \mathbf{0}_{m'} & I_{m'} \\ -{}^t I_{m'} & \mathbf{0}_{m'} \end{pmatrix}\right)({}^t g^{-1}) & \text{in case (C),} \end{cases}$$

on the  $GL(2m')$ -component, and

$$g \mapsto \begin{cases} \text{Ad}\left(\begin{pmatrix} \mathbf{0}_m & I_m \\ -{}^t I_m & \mathbf{0}_m \end{pmatrix}\right)({}^t g^{-1}) & \text{in case (B),} \\ \text{Ad}\left(\begin{pmatrix} & I_m \\ {}^t I_m & (-1)^m \end{pmatrix}\right)({}^t g^{-1}) & \text{in case (C),} \end{cases}$$

on the  $GL(2m)$ - or  $GL(2m + 1)$ -component. Thus we conclude:

(A) If  $E \neq F$ , we have

$$(A.5) \quad \widehat{H} = \widehat{L}^{s, \widehat{\theta}} = \{(d_{\mathbf{m}}(\underline{g}), \theta_{|\mathbf{m}|}(d_{\mathbf{m}}(\underline{g}))) \mid \underline{g} \in \prod_{i=1}^r GL(m_i, \mathbb{C})\}.$$

(B,C) If  $E = F$ ,  $\widehat{L}^{s, \widehat{\theta}}$  consists of the elements of the form

$$\text{diag}(d_{\mathbf{m}}(\underline{g}), \begin{pmatrix} a & b \\ & g \\ c & d \end{pmatrix}, \theta_{|\mathbf{m}|}(d_{\mathbf{m}}(\underline{g}))),$$

with  $\underline{g} \in \prod_{i=1}^r GL(m_i, \mathbb{C})$  and  $(g' := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, g)$  belongs to

$$\begin{cases} O(2m', \mathbb{C}) \times Sp(m, \mathbb{C}) & \text{in case (B),} \\ Sp(m', \mathbb{C}) \times O(2m+1, \mathbb{C}) & \text{in case (C).} \end{cases}$$

Hence

$$(A.6) \quad \widehat{H} \simeq \begin{cases} \prod_{i=1}^r GL(m_i, \mathbb{C}) \times SO(2m', \mathbb{C}) \times Sp(m, \mathbb{C}) & \text{in case (B),} \\ \prod_{i=1}^r GL(m_i, \mathbb{C}) \times Sp(m', \mathbb{C}) \times SO(2m+1, \mathbb{C}) & \text{in case (C).} \end{cases}$$

A.4.  $\text{Norm}(\widehat{H}, \widehat{L})$ . Our next task is to classify the  $L$ -action  $\rho_H$  for  $H$ . We always identify  $\mathcal{H}$  with  $\xi(\mathcal{H}) \subset {}^L L$ . Then a splitting  $\iota$  of  $\mathcal{H}$  (see A.1) can be written as

$$(A.7) \quad \iota(w) = a_\iota(w) \rtimes_{\rho_L} w, \quad w \in W_F,$$

where  $a_\iota(w)$  is an  $\widehat{L}$ -valued 1-cocycle satisfying  $(\text{Ad}(a_\iota(w)) \circ \rho_L(w))(\widehat{H}) = \widehat{H}$ . Thus we need to calculate  $\text{Norm}(\widehat{H}, \widehat{L})$ . For this we use the relation  $\text{Norm}(\widehat{H}, \widehat{L}) \subset \text{Out}(\widehat{H}) \ltimes (\widehat{H} \text{Cent}(\widehat{H}, \widehat{L}))$ .

Notice that  $\text{Cent}(\widehat{H}, \widehat{L})$  is contained in  $\text{Cent}(Z(\widehat{H}), \widehat{L})$  while this latter group equals  $\widehat{L}_{N_{\widehat{\theta}}(s)}$ . Thus  $\text{Cent}(\widehat{H}, \widehat{L}) = Z(\widehat{L}_{N_{\widehat{\theta}}(s)})$ .

Next calculate  $\text{Out}(\widehat{H})$ . We take the standard splitting  $\mathbf{spl}_{\widehat{H}} = (\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{Y}_\beta\})$  given by the “intersection” of  $\mathbf{spl}_{\widehat{L}}$  with  $\widehat{H}$ . That is,  $\mathcal{B}_H = \mathcal{B} \cap \widehat{H}$ ,  $\mathcal{T}_H = \mathcal{T}_{\widehat{\theta}}$  and

$$\mathcal{Y}_\beta := \sum_{\alpha; \alpha|_{\mathcal{T}_H} = \beta} \mathcal{X}_\alpha.$$

Then  $\text{Out}(\widehat{H})$  is identified with the subgroup of  $\text{Aut}(\widehat{H})$  which consists of elements preserving  $\mathbf{spl}_{\widehat{H}}$ . First we calculate the outer automorphism group for each direct component of  $\widehat{H}$ . Clearly  $Sp(m)$  and  $SO(2m+1)$  have no non-trivial outer automorphisms; we have only to consider  $GL(m)$  and  $SO(2m)$ .

Let us start with  $\text{Out}(GL(m))$ . Take  $\tau \in \text{Out}(GL(m))$ . Since  $\text{Aut}(\mathbb{G}_m) = \{\pm 1\}$ ,  $\tau|_{Z(GL(m))}$  and the automorphism  $\bar{\tau}$  of  $GL(m)/SL(m)$  induced by  $\tau$  coincide. On the other hand, we know from the Dynkin classification that  $\tau|_{SL(m)} \in \langle \theta_m \rangle$ . Now  $Z(GL(m)) \cap SL(m) = \mu_m$  implies that

$$(A.8) \quad \text{Out}(GL(m)) = \langle \theta_m \rangle$$

for  $m \geq 3$ . Here  $\mu_m$  denotes the finite algebraic group consisting of the  $m$ -th. roots of unity. When  $m = 2$ ,  $\text{Out}(SL(2))$  is trivial and  $\tau \in \text{Out}(GL(2))$  is determined by  $\bar{\tau}$ :  $\tau = \theta_2$  if  $\bar{\tau} = -1$ , and trivial if so is  $\bar{\tau}$ . Thus (A.8) is valid for any  $m \in \mathbb{N}$ .

Next comes  $SO(2m)$ . For  $SO(4)$ , we write

$$H := \{(g_1, g_2) \in GL(2)^2 \mid \det g_1 = \det g_2\}.$$

Then we have an isomorphism

$$H/\mathbb{G}_m \ni (g_1, g_2) \longmapsto g_1 \otimes \theta_2(g_2) \in SO(4)$$

in our realization of  $SO(4)$ . From the consideration above, we know that  $\text{Out}(GL(2)^2) = \langle \theta_2 \rangle^2 \rtimes \mathfrak{S}_2$  and hence  $\text{Out}(H) = \langle \theta_2 \times \theta_2 \rangle \times \mathfrak{S}_2$ . Here,  $\mathfrak{S}_n$  denotes the symmetric group of degree  $n$ . Since  $\theta_2 \times \theta_2$  induces the trivial automorphism on  $H/\mathbb{G}_m$ , we conclude that

$$\text{Out}(SO(4)) = \mathfrak{S}_2 =: \langle \epsilon_2 \rangle.$$

Consider  $SO(6)$ .  $GL(4)$  acts on  $W := \mathbb{G}_a^4$  by the standard representation. If we equip the vector space  $V := \bigwedge^2 W$  with the quadratic form

$$\wedge: V \otimes V \ni v \otimes w \longmapsto v \wedge w \in \bigwedge^4 W \simeq \mathbb{G}_a,$$

we obtain the homomorphism  $\varphi: SL(4) \rightarrow O(V)$ . If we write  $\{e_1, \dots, e_4\}$  for the standard basis of  $W$  and identify  $e_1 \wedge \dots \wedge e_4 \in \bigwedge^4 W$  with  $1 \in \mathbb{G}_a$ , then  $O(V)$  coincides with  $O(6)$  in our realization. A Witt basis of  $V$  is given by

$$\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3; e_1 \wedge e_4, e_2 \wedge e_4, e_3 \wedge e_4\}.$$

Thus  $\varphi$  restricted to the diagonal subgroup is given by

$$\varphi(\text{diag}(t_1, t_2, t_3, (t_1 t_2 t_3)^{-1})) = \text{diag}(t_1 t_2, t_1 t_3, t_2 t_3, (t_2 t_3)^{-1}, (t_1 t_3)^{-1}, (t_1 t_2)^{-1}).$$

This shows that  $\varphi$  descends to the isomorphism  $SL(4)/\mu_2 \xrightarrow{\sim} SO(6)$ , and we conclude

$$\text{Out}(SO(6)) = \langle \epsilon_3 \rangle,$$

where  $\epsilon_3$  corresponds to  $\theta_4 \in \text{Out}(SL(4)/\mu_2)$ . Now we consider general  $m$ . Write  $\epsilon_m$  for the unique outer automorphism of  $SO(2m)$  coming from  $\text{Int}(O(2m))$ . Then we have

$$(A.9) \quad \text{Out}(SO(2m)) = \langle \epsilon_m \rangle.$$

This follows from the Dynkin classification for  $m > 4$ . When  $m = 4$ , the automorphism group of the Dynkin diagram is  $\mathfrak{S}_3$  but only the trivial automorphism and  $\epsilon_4$  lift to automorphisms of  $SO(8)$  (or its based root datum). The case of  $m \leq 3$  is treated above.

We have proved the following.

LEMMA A.3: *If we write*

$$\widehat{H} \simeq \begin{cases} \prod_{i=1}^{\ell} GL(n_i)^{k_i} & \text{in case (A),} \\ \prod_{i=1}^{\ell} GL(n_i)^{k_i} \times SO(2m') \times Sp(m) & \text{in case (B),} \\ \prod_{i=1}^{\ell} GL(n_i)^{k_i} \times Sp(m') \times SO(2m+1) & \text{in case (C),} \end{cases}$$

with  $n_i \neq n_j$ , ( $i \neq j$ ), then we have

$$\text{Out}(\widehat{H}) = \begin{cases} \prod_{i=1}^{\ell} \langle \theta_{n_i} \rangle^{k_i} \rtimes \mathfrak{S}_{k_i} & \text{in cases (A) and (C),} \\ \prod_{i=1}^{\ell} \langle \theta_{n_i} \rangle^{k_i} \rtimes \mathfrak{S}_{k_i} \times \langle \epsilon_m \rangle & \text{in case (B).} \end{cases}$$

We have written  $\theta_0$  and  $\epsilon_0$  for the trivial automorphism of  $\{1\}$ .

COROLLARY A.4: *We have*

$$\text{Norm}(\widehat{H}, \widehat{L}) = \begin{cases} \prod_{i=1}^{\ell} \mathfrak{S}_{k_i} \ltimes (\widehat{H}Z(\widehat{L}_{N_{\theta}(s)})) & \text{in case (A),} \\ \text{Out}(\widehat{H}) \ltimes (\widehat{H}Z(\widehat{L}_{N_{\theta}(s)})) & \text{in cases (B), (C).} \end{cases}$$

A.5. ELLIPTICITY. Here we show that the ellipticity of  $(H, \mathcal{H}, s, \xi)$  eliminates many cumbersome cases.

First we examine the condition (A.1). Granting (A.7), this reads

$$\text{Ad}(s)(\widehat{\theta}(h \cdot a_{\iota}(w))) \rtimes_{\rho_L} w = h \cdot a_{\iota}(w)a'(w) \rtimes_{\rho_L} w$$

for  $h \in \widehat{H}$ ,  $w \in W_F$ . Here  $a'$  is a  $Z(\widehat{L})$ -valued 1-cocycle which is trivial if  $F$  is local, and everywhere locally trivial if  $F$  is global. We fix once and for all  $w_{\sigma} \in W_F \setminus W_E$  when  $E \neq F$ . Then (A.1) is equivalent to the following two formulae:

$$(A.10a) \quad \text{Ad}(s)\widehat{\theta}(a_{\iota}(w)) = a_{\iota}(w)a'(w), \quad w \in W_E,$$

$$(A.10b) \quad s\widehat{\theta}(a_{\iota}(w_{\sigma}))\rho_L(\sigma)(s)^{-1} = a_{\iota}(w_{\sigma})a'(w_{\sigma}).$$

Since  $a'|W_E$  is trivial or everywhere locally trivial if and only if  $a'|W_E = 1$ , (A.10a) becomes

$$(A.10a') \quad a_\iota(W_E) \subset \widehat{L}^{s, \widehat{\theta}}.$$

(A.10b) effects only case (A). Then  $a'$  is (everywhere locally) trivial if and only if  $a'(w_\sigma) = (z\mathbf{1}_n, z^{-1}\mathbf{1}_n)$  for some  $z \in \mathbb{C}^\times$ . Thus writing  $a_\iota(w_\sigma) = (x, y)$ ,  $x, y \in GL(n, \mathbb{C})$ , we have

$$(A.10b') \quad (\mathbf{1}_n, \theta_n(d_{\mathbf{m}}(\underline{\lambda}))) (\theta_n(y), \theta_n(x)) (\theta_n(d_{\mathbf{m}}(\underline{\lambda}))^{-1}, \mathbf{1}_n) = (zx, z^{-1}y).$$

A.5.1. *Ellipticity in case (A).* Since  $a_\iota(W_E) \subset \widehat{L}^{s, \widehat{\theta}} = \widehat{H}$  in this case, only  $w \in W_F \setminus W_E$  acts non-trivially on  $Z(\widehat{H})$  and we have

$$\rho_H(w_\sigma) d_{\mathbf{m}}(\underline{z}) = \text{Ad}(x) \theta_{|\mathbf{m}|}(d_{\mathbf{m}}(\underline{z})).$$

The ellipticity condition holds only if this equals  $d_{\mathbf{m}}(z_1^{-1}, \dots, z_r^{-1})$ , that is,  $x$  must be of the form

$$x = \begin{pmatrix} & & x_1 \\ & x_2 & \\ & \ddots & \\ x_r & & \end{pmatrix}, \quad x_i \in GL(m_i, \mathbb{C}).$$

Now (A.10b') gives

$$\begin{aligned} & z \begin{pmatrix} & & \lambda_r^{-1} \theta_{m_r}(x_r) \\ & \ddots & \\ \lambda_1^{-1} \theta_{m_1}(x_1) & & \end{pmatrix} = z \theta_n(d_{\mathbf{m}}(\underline{\lambda})) x = y \\ & = z^{-1} \theta_n(x) d_{\mathbf{m}}(\underline{\lambda}) = z^{-1} \begin{pmatrix} & & \lambda_r \theta_{m_r}(x_r) \\ & \ddots & \\ \lambda_1 \theta_{m_1}(x_1) & & \end{pmatrix}, \end{aligned}$$

and hence  $\lambda_i^2 = z^2$  for any  $1 \leq i \leq m$ . Since  $s$  is considered modulo  $Z(\widehat{L})_{\widehat{\theta}}$ , we may assume that  $z = \pm 1$  and

$$(A.11) \quad s = s_m := (\mathbf{1}_n, \text{diag}(\mathbf{1}_m, -\mathbf{1}_{m'})), \quad m + m' = n.$$

A.5.2. *Ellipticity in cases (B), (C).* Since  $\rho_L$  is trivial in these cases, the 1-cocycle  $a_\iota(w)$  is  $\text{Norm}(\widehat{H}, \widehat{L})$ -valued and we have

$$\rho_H(w)|Z(\widehat{H}) = \text{Ad}(a_\iota(w))|Z(\widehat{H}), \quad w \in W_F.$$



Also (A.10a') forces that  $a_\iota(W_F)$  is contained in

$$(A.12) \quad \mathrm{Im}(\widehat{L}^{s,\widehat{\theta}} \rightarrow \mathrm{Norm}(\widehat{H}, \widehat{L})) = \begin{cases} \langle \epsilon_m \rangle \rtimes \widehat{H} & \text{in case (B),} \\ \{\pm 1\} \times \widehat{H} & \text{in case (C).} \end{cases}$$

Here in case (C),  $\{\pm 1\}$  is the center of  $O(2m+1) \subset \widehat{L}^{s,\widehat{\theta}}$ . Then we can easily see that  $\xi(Z(\widehat{H})^{\Gamma_F})^0$  contains

$$\begin{cases} \{(z_i \mathbf{1}_{m_i})_{i=1}^r \times \pm \mathbf{1}_{2m'} \times \pm \mathbf{1}_{2m} \mid z_i \in \mathbb{C}^\times\} & \text{in case (B),} \\ \{(z_i \mathbf{1}_{m_i})_{i=1}^r \times \pm \mathbf{1}_{2m'} \times \mathbf{1}_{2m+1} \mid z_i \in \mathbb{C}^\times\} & \text{in case (C).} \end{cases}$$

Thus the ellipticity is equivalent to  $r = 0$  and we have

$$(A.13) \quad s = s_m := \begin{cases} \mathrm{diag}(\mathbf{1}_{m'}, \mathbf{1}_{2m}, -\mathbf{1}_{m'}) & \text{in case (B),} \\ \mathrm{diag}(\mathbf{1}_{m'}, \mathbf{1}_{2m+1}, -\mathbf{1}_{m'}) & \text{in case (C).} \end{cases}$$

A.6. ELLIPTIC ENDOSCOPIC GROUPS. Now we can classify  $H$ .

LEMMA A.5: *The elliptic endoscopic groups of  $(L, \theta, \mathbf{1})$  are the following.*

(A) *There is only one endoscopic group  $H_m = U_{E/F}(m) \times U_{E/F}(m')$  associated to  $s_m$  in (A.11). Here  $U_{E/F}(m)$  is the quasisplit unitary group in  $m$  variables attached to  $E/F$ .*

(B) *The endoscopic groups associated to  $s_m$  in case (B) of (A.13) are  $H_m = SO(2m') \times SO(2m+1)$ ,  ${}^K H_m = {}^K SO(2m') \times SO(2m+1)$ . Here  $SO(n)$  denotes the split special orthogonal group in  $n$  variables.  $K$  is a quadratic extension of  $F$ .  ${}^K SO(2m')$  is the quasisplit special orthogonal group such that*

(1)  ${}^K SO(2m') \otimes_F K \simeq SO(2m')_K$ .

(2) *The generator  $\sigma$  of  $\Gamma_{K/F} = \mathrm{Gal}(K/F)$  acts on  ${}^K SO(2m')$  by the unique element of  $\mathrm{Int}(O(2m')) \setminus \mathrm{Int}(SO(2m'))$  which preserves a splitting.*

(C) *There is only one endoscopic group  $H_m = SO(2m'+1) \times Sp(m)$  associated to  $s_m$  in case (C) of (A.13).*

*Proof:* To classify  $H$  or equivalently  ${}^L H = \widehat{H} \rtimes_{\rho_H} W_F$ , it suffices to determine the Galois action  $\rho_H$ . By the condition imposed on  $\mathcal{H}$  in A.1, it suffices to classify  $\{\mathrm{Ad}(a_\iota(w)) \circ \rho_L(w)\}_{w \in W_F}$  modulo  $\mathrm{Int}(\widehat{H})$ .

We begin with case (A) where  $s = s_m$  and  $\widehat{H} = GL(m, \mathbb{C}) \times GL(m', \mathbb{C})$ . We know from A.5.1 that  $a_\iota(W_E) \subset \widehat{H}_m$  and

$$a_\iota(w_\sigma) = \left( \begin{pmatrix} & x_1 \\ x_2 & \end{pmatrix}, \begin{pmatrix} & \mp \theta_{m'}(x_2) \\ \pm \theta_m(x_1) & \end{pmatrix} \right).$$

It follows that

$$\rho_{H_m}(w) = \begin{cases} \text{id}_{GL(m)} \times \text{id}_{GL(m')} & \text{if } w \in W_E, \\ \theta_m \times \theta_{m'} & \text{otherwise,} \end{cases}$$

and (A) is proved.

Next consider cases (B) and (C). For  $s = s_m$ , we know

$$\widehat{H}_m = \begin{cases} SO(2m', \mathbb{C}) \times Sp(m, \mathbb{C}) & \text{in case (B),} \\ Sp(m', \mathbb{C}) \times SO(2m+1, \mathbb{C}) & \text{in case (C),} \end{cases}$$

while  $a_\iota(w)$  is contained in (A.12). In case (C),  $\rho_{H_m}$  must be trivial and we have done. Consider case (B). If  $a_\iota(W_F) \subset \widehat{H}_m$ ,  $\rho_{H_m}$  is again trivial and we have  $H_m$ . But otherwise, we have the quadratic extension  $K$  determined by  $W_K := a_\iota^{-1}(\widehat{H}_m)$  and

$$a_\iota(w) \in \begin{cases} \widehat{H}_m & \text{if } w \in W_K, \\ \epsilon_m \widehat{H}_m & \text{otherwise.} \end{cases}$$

Thus we get  ${}^K H_m$ . ■

A.7. ELLIPTIC ENDOSCOPIC DATA. Finally we conclude the following.

**THEOREM A.6:** *The sets of elliptic endoscopic data for the triple  $(L, \theta, \mathbf{1})$  up to isomorphisms are the following. In all cases, we identify  $\widehat{H}$  and  $\mathcal{H}$  with their images under  $\xi$ .*

(A)  $E \neq F$ .  $\mathcal{E}_m = (H_m, {}^L H_m, s_m, \xi_m)$ ,  $0 \leq m \leq n$ . Here, writing  $m' := n - m$ ,

$$\begin{aligned} H_m &= U_{E/F}(m) \times U_{E/F}(m'), \quad s_m := (\mathbf{1}_n, \text{diag}(\mathbf{1}_m, -\mathbf{1}_{m'})), \\ \xi_m(h, h') &= \left( \begin{pmatrix} h & \\ & h' \end{pmatrix}, \begin{pmatrix} \theta_{m'}(h') & \\ & \theta_m(h) \end{pmatrix} \right), \quad (h, h') \in \widehat{H}_m, \\ \xi_m|_{W_E} &= (\text{diag}(\mathbf{1}_m, \mu \mathbf{1}_{m'}), \text{diag}(\mu^{-1} \mathbf{1}_{m'}, \mathbf{1}_m)) \rtimes_{\rho_L} \text{id}_{W_E}, \\ \xi_m(w_\sigma) &= \left( \begin{pmatrix} & \mathbf{1}_m \\ \mathbf{1}_{m'} & \end{pmatrix}, \begin{pmatrix} & -\mathbf{1}_{m'} \\ \mathbf{1}_m & \end{pmatrix} \right) \rtimes_{\rho_L} w_\sigma. \end{aligned}$$

$\mu$  is a character of  $E^\times$  (resp.  $\mathbb{A}_E^\times/E^\times$ ) whose restriction to  $F^\times$  (resp.  $\mathbb{A}_F^\times$ ) is the quadratic character  $\omega_{E/F}$  associated to  $E/F$  by the classfield theory if  $F$  is local (resp. global).  $w_\sigma$  is a fixed element in  $W_F \setminus W_E$ .

(B)  $E = F$  and  $n = 2n'$  is even.  $\mathcal{E}_m = (H_m, {}^L H_m, s_m, \xi_m)$ ,  $0 \leq m \leq n'$  and  ${}^K \mathcal{E}_m = ({}^K H_m, {}^L({}^K H_m), s_m, {}^K \xi_m)$ ,  $0 \leq m \leq n' - 1$  for each quadratic extension  $K$  of  $F$ . Here, writing  $m' = n' - m$ ,

$$\begin{aligned} H_m &= SO(2m') \times SO(2m+1), \\ {}^K H_m &= {}^K SO(2m') \times SO(2m+1), \quad s_m = \text{diag}(\mathbf{1}_{m'}, \mathbf{1}_{2m}, -\mathbf{1}_{m'}), \end{aligned}$$

$$\begin{aligned} \bullet \xi_m \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, h \right) &= \begin{pmatrix} a & & b \\ & h & \\ c & & d \end{pmatrix}, \quad (h' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, h) \in \widehat{H}_m = {}^K \widehat{H}_m, \\ \xi_m(w) &= \mathbf{1}_n \times w, \quad w \in W_F, \\ {}^K \xi_m(w) &= \begin{cases} \mathbf{1}_n \times w & \text{if } w \in W_K, \\ \left( \begin{pmatrix} & & I_{m'} \\ & \mathbf{1}_{2m} & \\ {}^t I_{m'} & & \end{pmatrix} \right) \times w & \text{otherwise.} \end{cases} \end{aligned}$$

$$\bullet \xi_m = \xi_m \text{ or } {}^K \xi_m.$$

(C)  $E = F$  and  $n = 2n' + 1$  is odd.  $\mathcal{E}_m = (H_m, {}^L H_m, s_m, \xi_m)$  and  ${}^K \mathcal{E}_m = (H_m, {}^L H_m, s_m, {}^K \xi_m)$  for each quadratic extension  $K$  of  $F$ ,  $0 \leq m \leq n'$ . Here, writing  $m' := n' - m$ ,

$$\begin{aligned} H_m &= SO(2m' + 1) \times Sp(m), \quad s_m = \text{diag}(\mathbf{1}_{m'}, \mathbf{1}_{2m+1}, -\mathbf{1}_{m'}), \\ \bullet \xi_m \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, h \right) &= \begin{pmatrix} a & & b \\ & h & \\ c & & d \end{pmatrix}, \quad (h' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, h) \in \widehat{H}_m, \\ \xi_m(w) &= \mathbf{1}_n \times w, \quad w \in W_F, \\ {}^K \xi_m(w) &= \begin{cases} \mathbf{1}_n \times w & \text{if } w \in W_K, \\ \text{diag}(\mathbf{1}_{m'}, -\mathbf{1}_{2m+1}, \mathbf{1}_{m'}) \times w & \text{otherwise.} \end{cases} \end{aligned}$$

$$\text{Again } \bullet \xi_m = \xi_m \text{ or } {}^K \xi_m.$$

*Proof:* First note that the sets of data listed in the theorem are all well-defined. Also, it follows easily from the above arguments that any sets of elliptic endoscopic data for  $(L, \theta, 1)$  is isomorphic to one in this list. (Observe that the isomorphism class of  $\xi_m$  in case (A) is independent of the choice of  $\mu$ .) Thus what is left to show is that the data in the theorem are not isomorphic to each other. This is obvious in cases (A) and (B): in case (A),  $\xi_{m_1}(w)$  and  $\xi_{m_2}(w)$  are not  $\widehat{L}$ -conjugate unless  $m_1 = m_2$ ; in case (B), the endoscopic groups are all distinct. In case (C), it suffices to show that  $\mathcal{E}_m$  and  ${}^K \mathcal{E}_m$  are not isomorphic. Suppose that these are isomorphic. Then an isomorphism  $g \in \widehat{L}$  from  $\mathcal{E}_m$  to  ${}^K \mathcal{E}_m$  belongs to  $\text{Norm}(\widehat{H}, \widehat{L}) = \widehat{H} \text{Cent}(\widehat{H}, \widehat{L})$ . But such a  $g$  centralizes  ${}^K \xi_m|_{W_F}$  and  $\xi_m|_{W_F}$ . ■

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